Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH2617-WE01

Title: Elementary Number Theory II

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **TWO** answers from Section A and the best **TWO** answers from Section B. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A.

Revision:	

SECTION A

1. (i) Give the definition of a multiplicative function $f : \mathbb{N} \to \mathbb{N}$.

   (ii) Let $\phi(n)$ be the Euler totient function. Is the function $f(n) = n \cdot \phi(n)$ multiplicative? Justify your answer.

   (iii) Is the function $g(n)$ given by the formula

$$g(n) = \begin{cases} -1 & \text{if } n \text{ is prime} \\ 1 & \text{otherwise} \end{cases}$$

   multiplicative?

   (iv) Compute $\phi(2775)$.

2. (i) Let m_1 and m_2 be two positive integers which can be written as a sum of two squares. Prove that their product $m_1 m_2$ is also a sum of two squares. (You are not allowed just to quote the corresponding lemma from the lectures. You need to prove it.)

   (ii) You are given that 2017 is a prime number. How many different representations as a sum of two squares does 2017 have? (Two representations $2017 = x^2 + y^2$ and $2017 = y^2 + x^2$ are considered to be the same).

   (iii) Find two different representations of $2329 = 137 \cdot 17$ as a sum of two squares.

3. (i) State Fermat's Little Theorem.

   (ii) You are given that $2^{133333337} \equiv 7332991 \pmod{133333337}$. Can you conclude that 133333337 is prime? That 133333337 is composite? Justify your answer.

   (iii) You are given that 1997 is prime. Compute an integer n such that $0 \leq n < 1997$ and $2^{4000} \equiv n \pmod{1997}$.

   (iv) You are given that $11^{1728} \equiv 1 \pmod{1729}$. Is 1729 a prime number? If yes, justify your answer. If no, provide a prime factorisation of 1729.

SECTION B

4. (a) Let $p > 3$ be a prime number. Show that the quadratic congruence

$$x^2 + x + 1 \equiv 0 \pmod{p}$$

has an integer solution if and only if -3 is a quadratic residue modulo p.

(b) Show that for any $n \in \mathbb{N}$ all prime divisors of a number $n^2 + n + 1$ are either equal to 3 or of the form $3k + 1$ with positive integer k.

(c) Show that there are infinitely many prime numbers of the form $3k + 1$ with positive integer k.

(d) Prove the following statement: if $2^{3n} - 1$ is divisible by 131 then $2^n - 1$ is divisible by 131.

5. (a) Give the definition and the general formula of a primitive Pythagorean triple.

(b) Prove that for any primitive Pythagorean triple (x, y, z) the product xyz is always divisible by 60. [Hint: you can independently check that it is divisible by 3, 4 and 5.]

(c) Find all primitive Pythagorean triples (x, y, z) such that $y = 2x + 1$ and $y < 1000$.

6. (a) Find the continued fraction of $\sqrt{34}$.

(b) Find a rational number p/q such that $p, q \in \mathbb{N}$, $q < 100$ and

$$\left| \sqrt{34} - \frac{p}{q} \right| < \frac{1}{2 \cdot 10^4}.$$

(c) Find two different solutions of the equation

$$x^2 - 34y^2 = 1. \qquad (*)$$

(d) Let $(x_0, y_0) \in \mathbb{N}^2$ be a solution of the equation $(*)$. Show that an infinite series (x_n, y_n) of solutions of $(*)$ can be achieved by the following recurrent formula:

$$x_{n+1} = 35x_n + 204y_n, \quad y_{n+1} = 6x_n + 35y_n.$$

[Hint: using quadratic field extensions may help here.]