Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH2647-WE01

Title: Probability II

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **TWO** answers from Section A and the best **TWO** answers from Section B. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. A die showing '1','2','3','4','5','6' with corresponding positive probabilities $p_1, p_2, p_3, p_4, p_5, p_6$ is tossed repeatedly, with individual outcomes assumed independent.

   (a) Find $\mathsf{P}(B_k)$, where $B_k = \big\{\text{no '1' from toss } k \text{ onwards}\big\}$.

   (b) Show that $B = \big\{\text{finitely many '1' observed}\big\} = \cup_{k\geq 1} B_k$, and hence find $\mathsf{P}(B)$.

   (c) Deduce the value of $\mathsf{P}(A)$ with $A = \big\{\text{infinitely many '1' observed}\big\}$.

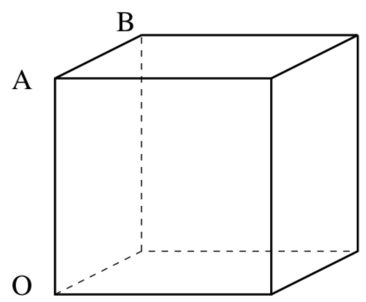
   (d) Let H_k be the event that '1' was observed on the kth toss and '2' was observed on the $(k+1)$th toss. Use the Borel–Cantelli lemma to find the probability of the event $\big\{H_k \text{ occurs infinitely often}\big\}$.

   You should give a clear statement of any result you use.

2. A random walker moves on the vertices of a cube. Two distinguishable vertices—A and B in the picture below—are absorbing (once entering A or B the walker stays there forever), whereas from every other vertex the walker jumps on the next move to one of its three neighbours with equal probabilities.

   (a) If the walker starts at O, find the probability h_A that it gets absorbed in A. Similarly, find the absorption probability h_B in B. Explain your findings.

   (b) If the walker starts at O, find the expected number of jumps until absorption in $\{A, B\}$.



3. Carefully state the Monotone Convergence Theorem and the Dominated Convergence Theorem.

   Let X be a random variable with $\mathsf{E}|X| < \infty$. For some positive α define

$$Y_N \stackrel{\mathsf{def}}{=} X \wedge N^\alpha \equiv \begin{cases} X\,, & \text{if } X \leq N^\alpha\,, \\ N^\alpha\,, & \text{if } X > N^\alpha\,, \end{cases} \qquad Z_N \stackrel{\mathsf{def}}{=} X \mathbb{1}_{|X| \leq N^\alpha} \equiv \begin{cases} X\,, & \text{if } |X| \leq N^\alpha\,, \\ 0\,, & \text{if } |X| > N^\alpha\,. \end{cases}$$

   (a) By using an appropriate limit result, show that

$$\mathsf{E}\big(Y_N\big) \to \mathsf{E}(X) \qquad \text{and} \qquad \mathsf{E}\big(Z_N\big) \to \mathsf{E}(X) \qquad \text{as } N \to \infty\,.$$

   (b) Find the limit of $\mathsf{E}\big(X \mathbb{1}_{|X| > N^\alpha}\big)$ as $N \to \infty$.

SECTION B

4. Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/16 & 0 & 15/16 \\ 0 & 0 & 0 & 0 & 1/3 & 2/3 \end{pmatrix}.$$

(a) Find all stationary distributions for this chain.

(b) Identify classes of communicating states and explain which classes are closed and which are not.

(c) Find the n-step transition probabilities $p_{ii}^{(n)}$, $i = 1, 2, 3, 4$, and use these values to check which states are transient and which are recurrent.

(d) Determine the period of every state.

In your answer you should give a clear statement of any result you use.

5. Let X_1, X_2, $\dots$, X_n be independent identically distributed random variables with values in $\{0, 1, 2, \dots\}$ and let $N \geq 0$ be an integer-valued random variable independent of $\{X_k\}_{k \geq 1}$. Let $G_X(s)$ be the generating function for X_1 and $G_N(s)$ be the generating function for N.

(a) Prove that $S_N \stackrel{\mathsf{def}}{=} X_1 + \dots + X_N$ has generating function

$$G_{S_N}(s) = G_N\big(G_X(s)\big).$$

(b) Find the expectation of S_N in terms of $\mathsf{E}X_1$ and $\mathsf{E}N$, assuming that the latter two expectations are finite.

(c) A fair die (showing '1','2','3','4','5','6') is tossed repeatedly (with individual outcomes assumed independent) up to and including the moment when it shows '6' for the first time. Find the expected total score which was seen up to this moment.

6. Define carefully the notions of convergence in probability, convergence in L^p, $p > 0$, and convergence almost surely.

   Let X, $(X_n)_{n\geq 1}$, Y, $(Y_n)_{n\geq 1}$ be some random variables, and let c be some real number.

   (a) If X_n converges to X in probability (as $n \to \infty$) and Y_n converges to Y in probability (as $n \to \infty$), show that $X_n + Y_n$ converges to $X + Y$ in probability (as $n \to \infty$).

   (b) If X_n converges to X in probability (as $n \to \infty$) and $c \neq 0$, show that cX_n converges to cX in probability (as $n \to \infty$).

   (c) If X_n converges to X almost surely (as $n \to \infty$) and Y_n converges to Y almost surely (as $n \to \infty$), show that $X_n + Y_n$ converges to $X + Y$ almost surely (as $n \to \infty$).

   For each of the following statements, prove the claim or give a counter-example:

   (d) If $X_n \to X$ as $n \to \infty$ in probability, then $X_n \to X$ as $n \to \infty$ in L^p, for some $p > 0$.

   (e) If $X_n \to X$ as $n \to \infty$ in L^p, for some $p > 0$, then $X_n \to X$ as $n \to \infty$ almost surely.