Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH2657-WE01

Title:

Special Relativity & Electromagnetism II

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **TWO** answers from Section A and the best **TWO** answers from Section B. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. (a) Faraday's law in integral form states that

$$\oint_C \mathbf{E} \cdot \mathbf{dx} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dA} \,,$$

where the closed curve C is the boundary of the arbitrary surface S. Use this to derive one of Maxwell's differential equations.

(b) Gauss' law in electrostatics states that $\oint_S \epsilon_0 E_i dA_i = \int_V \rho \, dV$ where S is the boundary of the arbitrary volume V. Use this to compute the electric field at distance r from the centre of a spherically symmetric charge distribution with density

$$\rho = \frac{Q}{2\pi r(1+r^2)^2}$$

(You may assume the electric field points outwards from the origin).

2. A muon is an unstable particle. In its rest-frame it has a lifetime of 2×10^{-6} s. A muon is created in the atmosphere 800 metres above the surface of the Earth.

(a) What is the minimum speed it should travel at to reach sea-level. (Take $c = 3 \times 10^8 \; ms^{-1}$.)

(b) If it travels to sea-level at this speed how long is its lifetime in the rest-frame of the laboratory? In the rest-frame of the muon, what length of atmosphere does it travel through?

3. In one inertial frame three events A, B and C have co-ordinates $(x_A^\mu) = (1, 1, 1, 1)$, $(x_B^\mu) = (14, 4, 5, 13)$ and $(x_C^\mu) = (13, 4, 5, 13)$.

(a) For each pair of events in turn state whether they are space-like, time-like or null separated. What can you conclude about the ordering of each pair of events in time?

(b) Two events occur in the same place in a certain inertial frame, and are separated by a time interval of 3 seconds. What is the spatial separation between these two events in an inertial frame in which the time separation is 5 seconds?

SECTION B

4. Maxwell's equations in empty space are

$$\partial_j E_j = 0, \quad \epsilon_{jkl}\partial_k E_l = -\dot{B}_j, \quad \partial_j B_j = 0, \quad \epsilon_{jkl}\partial_k B_l = \epsilon_0\mu_0 \dot{E}_j\,.$$

Define the complex vector field $\mathbf{F}$ to have components $F_j = E_j + icB_j$ with $c = 1/\sqrt{\epsilon_0\mu_0}$.

(a) Use Maxwell's equations in empty space to show that

$$\partial_j F_j = 0, \quad \epsilon_{jkl}\partial_k F_l = \frac{i}{c}\dot{F}_j$$

(b) Use these equations for $\mathbf{F}$ to show that $\mathbf{F}$ satisfies the wave-equation.

(c) Show that the equations for $\mathbf{F}$ are unchanged under the transformation $\mathbf{F} \to \mathbf{F}' = e^{i\theta}\mathbf{F}$ where θ is a constant. How do $\mathbf{E}$ and $\mathbf{B}$ change under this transformation?

5. The transformation

$$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu$$

is a Lorentz transformation if

$$\mathbf{L}^t \eta\, \mathbf{L} = \eta, \quad \text{with} \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(a) Show that this condition is satisfied by

$$(L^\mu{}_\nu) = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \mathbf{L}(\theta)\,,$$

and show how θ is related to the velocity of one frame relative to the other.

(b) Find the relation between θ_1, θ_2 and θ_3 so that

$$\mathbf{L}(\theta_2)\,\mathbf{L}(\theta_1) = \mathbf{L}(\theta_3)\,.$$

(c) Use the result in the last part to derive the rule for velocity addition in special relativity. Two particles are travelling away from each other, one along the positive x-axis, the other along the negative x-axis of an inertial frame in which each has speed $c/2$. What speed does one particle appear to have in the rest-frame of the other?

6. In an inertial frame F two particles with rest-mass m move along the x-axis from opposite directions but with the same speed and collide head on. They fuse to form a single particle with rest-mass M which then decays to produce two particles of mass m moving in opposite directions along the y-axis. The two initial particles have four-momenta p_1^μ and p_2^μ and the final two particles have four-momenta q_1^μ and q_2^μ.

   (a) Show that $M \geq 2m$ and that $p_1 \cdot p_2 = q_1 \cdot q_2$.

   (b) Show that $p_1 \cdot q_1 = p_1 \cdot q_2 = p_2 \cdot q_1 = p_2 \cdot q_2$

   (c) Suppose that the initial particles are photons, so that $m = 0$ and the frequency of the associated electromagnetic wave is given by the photon's energy divided by Planck's constant $\nu = E/h$.

   Suppose also that an observer is situated on the x-axis of F at $x = \infty$ moving with a velocity half that of light in the positive x-direction. Use the relativistic Doppler effect to find the frequency the observer measures of the electromagnetic wave associated with the photon moving towards the observer in terms of M, h and c.