

## **EXAMINATION PAPER**

Examination Session: May

2017

Year:

Exam Code:

MATH2657-WE01

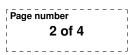
Title:

## Special Relativity & Electromagnetism II

Time Allowed:	2 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for the best <b>TWO</b> and the best <b>TWO</b> answers from Sec Questions in Section B carry <b>ONE an</b> marks as those in Section A.	tion B.	
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**Revision:** 





## SECTION A

1. (a) Faraday's law in integral form states that

$$\oint_C \mathbf{E} \cdot \mathbf{dx} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dA} \,,$$

where the closed curve C is the boundary of the arbitrary surface S. Use this to derive one of Maxwell's differential equations.

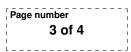
(b) Gauss' law in electrostatics states that  $\oint_S \epsilon_0 E_i dA_i = \int_V \rho \, dV$  where S is the boundary of the arbitrary volume V. Use this to compute the electric field at distance r from the centre of a spherically symmetric charge distribution with density

$$\rho = \frac{Q}{2\pi r (1+r^2)^2}$$

(You may assume the electric field points outwards from the origin).

- 2. A muon is an unstable particle. In its rest-frame it has a lifetime of  $2 \times 10^{-6}$  s. A muon is created in the atmosphere 800 metres above the surface of the Earth.
  - (a) What is the minimum speed it should travel at to reach sea-level. (Take  $c = 3 \times 10^8 \ ms^{-1}$ .)
  - (b) If it travels to sea-level at this speed how long is its lifetime in the rest-frame of the laboratory? In the rest-frame of the muon, what length of atmosphere does it travel through?

- 3. In one inertial frame three events A, B and C have co-ordinates  $(x_A^{\mu}) = (1, 1, 1, 1), (x_B^{\mu}) = (14, 4, 5, 13)$  and  $(x_C^{\mu}) = (13, 4, 5, 13).$ 
  - (a) For each pair of events in turn state whether they are space-like, time-like or null separated. What can you conclude about the ordering of each pair of events in time?
  - (b) Two events occur in the same place in a certain inertial frame, and are separated by a time interval of 3 seconds. What is the spatial separation between these two events in an inertial frame in which the time separation is 5 seconds?



## SECTION B

4. Maxwell's equations in empty space are

$$\partial_j E_j = 0, \quad \epsilon_{jkl} \partial_k E_l = -\dot{B}_j, \quad \partial_j B_j = 0, \quad \epsilon_{jkl} \partial_k B_l = \epsilon_0 \mu_0 \dot{E}_j.$$

Define the complex vector field **F** to have components  $F_j = E_j + icB_j$  with  $c = 1/\sqrt{\epsilon_0\mu_0}$ .

(a) Use Maxwell's equations in empty space to show that

$$\partial_j F_j = 0, \quad \epsilon_{jkl} \partial_k F_l = \frac{i}{c} \dot{F}_j$$

- (b) Use these equations for  $\mathbf{F}$  to show that  $\mathbf{F}$  satisfies the wave-equation.
- (c) Show that the equations for  $\mathbf{F}$  are unchanged under the transformation  $\mathbf{F} \rightarrow \mathbf{F}' = e^{i\theta}\mathbf{F}$  where  $\theta$  is a constant. How do  $\mathbf{E}$  and  $\mathbf{B}$  change under this transformation?

5. The transformation

$$x^{\mu} \rightarrow x^{\prime \mu} = L^{\mu}_{\ \nu} x^{\nu}$$

is a Lorentz transformation if

$$\mathbf{L}^{t} \eta \, \mathbf{L} = \eta, \quad \text{with} \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \,.$$

(a) Show that this condition is satisfied by

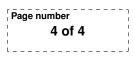
$$(L^{\mu}_{\nu}) = \begin{pmatrix} \cosh \theta & -\sinh \theta & 0 & 0 \\ -\sinh \theta & \cosh \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \mathbf{L}(\theta) \,,$$

and show how  $\theta$  is related to the velocity of one frame relative to the other.

(b) Find the relation between  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  so that

$$\mathbf{L}(\theta_2) \, \mathbf{L}(\theta_1) = \mathbf{L}(\theta_3) \, .$$

(c) Use the result in the last part to derive the rule for velocity addition in special relativity. Two particles are travelling away from each other, one along the positive x-axis, the other along the negative x-axis of an inertial frame in which each has speed c/2. What speed does one particle appear to have in the rest-frame of the other?



- 6. In an inertial frame F two particles with rest-mass m move along the x-axis from opposite directions but with the same speed and collide head on. They fuse to form a single particle with rest-mass M which then decays to produce two particles of mass m moving in opposite directions along the y-axis. The two initial particles have four-momenta  $p_1^{\mu}$  and  $p_2^{\mu}$  and the final two particles have four-momenta  $q_1^{\mu}$  and  $q_2^{\mu}$ .
  - (a) Show that  $M \ge 2m$  and that  $p_1 \cdot p_2 = q_1 \cdot q_2$ .
  - (b) Show that  $p_1 \cdot q_1 = p_1 \cdot q_2 = p_2 \cdot q_1 = p_2 \cdot q_2$
  - (c) Suppose that the initial particles are photons, so that m = 0 and the frequency of the associated electromagnetic wave is given by the photon's energy divided by Planck's constant  $\nu = E/h$ .

Suppose also that an observer is situated on the x-axis of F at  $x = \infty$  moving with a velocity half that of light in the positive x-direction. Use the relativistic Doppler effect to find the frequency the observer measures of the electromagnetic wave associated with the photon moving towards the observer in terms of M, h and c.