EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH2667-WE01

Title:
Monte Carlo II

Time Allowed:	1 hour 30 minutes	
Additional Material provided:	Tables: Normal.	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **ONE** answer from Section B. Questions in Section B carry **THREE TIMES** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. State the *inverse transform* algorithm for sampling from a continuous probability distribution and prove that the algorithm succeeds in sampling from the target distribution.

   Let the random variable X have probability density function

   $$f(x) = \begin{cases} \frac{1}{2}\cos x & \text{for } x \in [-\pi/2, \pi/2] \\ 0 & \text{otherwise} \end{cases}$$

   and suppose that 0.65 and 0.20 are two values sampled from Uniform$(0,1)$, the uniform distribution on $[0,1)$. Sample two values from the distribution of X.

2. Explain how to construct a random vector $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ having a bivariate normal distribution from a random vector $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ where Z_1 and Z_2 are independent standard normal, $\mathrm{N}(0,1)$, random variables. Show how the means and variances of X_1 and X_2, and the covariance between X_1 and X_2, depend on the elements of the construction.

   Let the mean vector and variance-covariance matrix of $\boldsymbol{X}$ be respectively $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 & 3.6 \\ 3.6 & 9 \end{pmatrix}$. Suppose that 1.4 and -0.7 are two values sampled from the standard normal distribution. Sample a value for $\boldsymbol{X}$.

3. Explain what is meant by the phrase "pseudo-random number generator".

   The general form of the linear congruential generator (LCG) is

   $$X_{i+1} = (aX_i + c) \mod M$$

   Explain how an LCG may be used to provide a source of pseudo-random numbers U_i approximately uniformly distributed in $[0,1]$. Why are the values not truly uniformly distributed?

   Consider the LCG having $a = 1365$, $c = 1$ and $M = 2^{11}$. Show that $3X_{i+1} + X_i - 3$ is always an integer multiple of 2^{11}. What is the implication for the distribution of pairs of successive values from the generator? Sketch a plot of U_{i+1} versus U_i.

SECTION B

4. (a) State the *acceptance-rejection* algorithm for sampling from a continuous *target* probability distribution using values sampled from another continuous *proposal* probability distribution. Make clear where the acceptance and rejection takes place. State clearly any conditions needed on the two distributions.

   (b) Prove that the algorithm succeeds in sampling from the target distribution.

   (c) The standard normal distribution, $\mathrm{N}(0,1)$, has probability density function $f(x) = (2\pi)^{-1/2}\exp(-\frac{1}{2}x^2)$ for real x. The standard double exponential distribution has probability density function $h(x) = \frac{1}{2}\exp(-|x|)$ for real x. Show that it is possible to use $h(x)$ as the proposal distribution when sampling from the standard normal distribution using the acceptance-rejection algorithm.

   (d) Another distribution has probability density function $g(x) = 1/\left[\pi\sqrt{2}(1+x^2/2)\right]$ for real x. Show that this distribution can also be used as the proposal distribution when sampling from the standard normal distribution using the acceptance-rejection algorithm.

   (e) Which is more efficient: sampling proposals from $h(x)$ or from $g(x)$?

5. (a) For a homogeneous Poisson process with rate λ, show that the distribution of the time, X_1, to the first event is $\mathrm{Exp}(\lambda)$. Show also that the time, X_2, between the first and second events has the same distribution and is independent of X_1. You may assume that the number of events, $N(t)$, up to time t has the Poisson distribution with mean λt and should state carefully any parts of the definition of the Poisson process that you use.

   (b) Suppose that 0.90, 0.33 and 0.54 are three values sampled from $\mathrm{Uniform}(0,1)$, the uniform distribution on $[0,1)$.

   Sample the first three event times from a Poisson process with rate $\lambda = 1/2$.

   (c) Using the same values sampled from $\mathrm{Uniform}(0,1)$ as in part (b), sample the first three event times from a non-homogeneous Poisson process with rate function $\lambda(t) = 2\sqrt{t}$ for $t > 0$.

   (d) A *tandem queue* is a single queue system with two servers where each customer must be served first by server 1 and then by server 2. Each server can serve only one customer at a time: a customer who has been served by server 1 may have to wait to be served by server 2. Customers will be served by both servers in the order in which they arrive. Suppose that:

   - customers arrive in the queue according to a Poisson process with rate λ;
   - the time taken by server 1 to serve a customer is random with cumulative distribution function $F_S(s)$;
   - the time taken by server 2 to serve a customer is random with cumulative distribution function $F_{S_2}(s)$.

   On the next page of the exam paper, an algorithm is provided for simulating the single-server, single queue model with Poisson process arrivals and service times following cumulative distribution function $F_S(s)$. Explain how the algorithm should be modified to simulate the tandem queue model. If you prefer, you may instead draw a flow-chart or write Python code or pseudo-code.

Single-server, single queue algorithm:

State of system:

- t is the current time, starts at 0;
- t_A is the time of the next arrival;
- t_S is the time when the current service ends, $t_S = \infty$ when the server is idle;
- Q is the number of people/items in the queue waiting to be served.

Simulation process:

1. sample X from $\text{Exp}(\lambda)$
   $t_A \leftarrow X$
   $t_S \leftarrow \infty$
   $Q \leftarrow 0$
   $t \leftarrow 0$
2. $t \leftarrow \min(t_A, t_S)$
3. If $t > t_{\max}$ **STOP**
4. If $t = t_A$
       record "arrival" event at time t
       $Q \leftarrow Q + 1$
       sample X from $\text{Exp}(\lambda)$
       $t_A \leftarrow t_A + X$
       if $t_S < \infty$
           go to step 2
   else
       record "service ended" event at time t
       if $Q = 0$
           $t_S \leftarrow \infty$
           go to step 2
5. record "service started" event at time t
   sample service time S from F_S
   $t_S \leftarrow t + S$
   $Q \leftarrow Q - 1$
   go to step 2