Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH3021-WE01

Title: Differential Geometry III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. Let $\boldsymbol{\alpha}$ be a plane curve given by

$$\boldsymbol{\alpha}(s) = (s^2, \frac{1}{s^2}),\ s > 0.$$

   (a) Show that $\boldsymbol{\alpha}$ is a regular curve.

   (b) Calculate the curvature of $\boldsymbol{\alpha}$.

   (c) Find the vertices of $\boldsymbol{\alpha}$.

2. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a plane regular curve. Here I is an open interval in $\mathbb{R}$.

   (a) Define the centre of curvature of $\boldsymbol{\alpha}$ at $u \in I$.

   (b) Let κ be the curvature of $\boldsymbol{\alpha}$, and suppose that $\kappa(u_0) \neq 0$. Denote by $\boldsymbol{p}$ the centre of curvature of $\boldsymbol{\alpha}$ at u_0, and define a function $g : I \to \mathbb{R}$ by

$$g(u) = \|\boldsymbol{\alpha}(u) - \boldsymbol{p}\|^2$$

   Show that $g'(u_0) = g''(u_0) = 0$.

   (c) Define the evolute of $\boldsymbol{\alpha}$. Find the evolute of the parabola $\boldsymbol{\alpha}(u) = (u, u^2)$, $u \in \mathbb{R}$.

3. Let $S = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$ be a cylinder.

   (a) Let $\theta_0 \in (0, \pi/2)$ be given. Find a regular curve $\boldsymbol{\alpha} : \mathbb{R} \to S$ through the point $(1, 0, 0) \in S$ forming the constant angle θ_0 with all parallels of S.

   (b) Assuming the curve $\boldsymbol{\alpha}$ from part (a) is parametrised by arc length, show that $\boldsymbol{\alpha}$ is a geodesic.

4. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a regular curve parametrised by arc length with nowhere vanishing curvature. Here I is an open interval in $\mathbb{R}$. The *normal plane* through $\boldsymbol{\alpha}(u)$ is the plane spanned by the unit normal and binormal vectors $\boldsymbol{n}(u)$ and $\boldsymbol{b}(u)$. Assume all normal planes of $\boldsymbol{\alpha}$ pass through a point $\boldsymbol{p}_0 \in \mathbb{R}^3$.

   (a) Show that $\boldsymbol{\alpha}$ lies in a sphere centred at $\boldsymbol{p}_0$.

   (b) Express the coefficients $\lambda(u), \mu(u), \nu(u)$ in the decomposition

$$\boldsymbol{\alpha}(u) - \boldsymbol{p}_0 = \lambda(u)\boldsymbol{t}(u) + \mu(u)\boldsymbol{n}(u) + \nu(u)\boldsymbol{b}(u)$$

   in terms of the curvature $\kappa(u)$, the torsion $\tau(u)$ and their derivatives. (Here $\boldsymbol{t}(u)$ denotes the unit tangent vector.)

5. Let $\boldsymbol{x} \colon \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrisation of a surface S in $\mathbb{R}^3$.

   (a) Define the Gauss map $\boldsymbol{N}$ of S.

   (b) Calculate the Gauss map of $\boldsymbol{x}(u, v) = (u, v, (u + v)^2)$.

   (c) Determine the image of the Gauss map of $\boldsymbol{x}$ defined as in (b).

6. (a) Define the Weingarten map and the second fundamental form of a surface $S \subset \mathbb{R}^3$ with local parametrisation $\boldsymbol{x}(u, v)$, $(u, v) \in U \subset \mathbb{R}^2$.

(b) Let a surface S be the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$. Consider the parametrisation of S by

$$\boldsymbol{x}(u, v) = (u, v, f(u, v)).$$

Find necessary and sufficient conditions for the second fundamental form of S to have the matrix

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

at a point $\boldsymbol{x}(u_0, v_0)$ with respect to the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$.

(c) A parametrisation of S is *isothermal* if the coefficients E, F, G of the first fundamental form satisfy conditions $E = G$ and $F = 0$ at every point. Show that if $\boldsymbol{x} : U \to S$ is an isothermal parametrisation, then

$$\boldsymbol{x}_{uu}(u, v) + \boldsymbol{x}_{vv}(u, v) = 2E(u, v)H(u, v)\boldsymbol{N}(\boldsymbol{x}(u, v)),$$

where H is the mean curvature and $\boldsymbol{N}$ is the Gauss map.

SECTION B

7. Let $S^2(1) = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ be a 2-dimensional unit sphere. For $(u, v) \in \mathbb{R}^2$, let $\boldsymbol{x}(u, v)$ be the point of intersection of the line in $\mathbb{R}^3$ through $(u, v, 0)$ and $(0, 0, 1)$ with $S^2(1)$ (different from $(0, 0, 1)$). $\boldsymbol{x}$ is called a *stereographic projection* of the unit sphere.

   (a) Find an explicit formula for $\boldsymbol{x}(u, v)$.

   (b) Let P be the plane given by $\{z = 1\}$, and for $(x, y, z) \in \mathbb{R}^3 \setminus P$, let $\boldsymbol{F}(x, y, z) \in \mathbb{R}^2$ be such that $(\boldsymbol{F}(x, y, z), 0) \in \mathbb{R}^3$ is the intersection with the (x, y)-plane of the line through $(0, 0, 1)$ and (x, y, z). Show that

   $$\boldsymbol{F}(x, y, z) = \frac{1}{1 - z}(x, y).$$

   (c) Show that $\boldsymbol{F} \circ \boldsymbol{x} = \mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$ and deduce that $\boldsymbol{x}$ is a local parametrisation of $S^2(1) \setminus \{(0, 0, 1)\}$.

   (d) Show that $\boldsymbol{x}$ is conformal.

8. (a) Let $\boldsymbol{\alpha}$ be a regular curve contained in a surface $S \subset \mathbb{R}^3$. Define its normal curvature κ_n.

   (b) Assume that S has positive Gauss curvature. Show that the normal curvature κ_n of $\boldsymbol{\alpha}$ is nowhere vanishing.

   (c) Assume that S has negative Gauss curvature. Let $p \in S$. Show that there exists a parametrised curve $\boldsymbol{\alpha}$ in S containing p which has positive normal curvature κ_n at p.

9. (a) Give the definition of a minimal surface in $\mathbb{R}^3$.

   (b) Show that the catenoid $\boldsymbol{x}(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$, where $u \in (0, 2\pi)$, $v \in \mathbb{R}$ and $a > 0$ is a constant, is a minimal surface in $\mathbb{R}^3$.

   (c) Assume that $S \subset \mathbb{R}^3$ is a minimal surface with nowhere vanishing Gauss curvature. Show that the Gauss map on S is conformal and find the conformal factor.

10. (a) State the global Gauss–Bonnet theorem explaining all the notations which you use.

    (b) Let $S \subset \mathbb{R}^3$ be a ruled surface. Show that the Gauss curvature K of S is nowhere positive.

    (c) Let S be as in (b) and assume, in addition, that S is homeomorphic to the plane and $K \neq 0$ in any point of S. Show that a geodesic on S cannot be self-intersecting.

    (d) Under the assumptions of (c), show that two different geodesics cannot have more than one point of intersection.

