Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH3041-WE01

Title:

Galois Theory III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. Find the minimal polynomial for $\sqrt[6]{3}+1$ over:

   (a) $\mathbb{Q}(\sqrt{3})$;

   (b) $\mathbb{Q}(\sqrt[3]{3})$.

2. Find splitting fields and Galois groups of the following polynomials over $\mathbb{Q}$:

   (a) $T^4+2T^3+2T^2$;

   (b) T^6-9.

3. Suppose L/K is a finite field extension.

   (a) Prove that any $\theta \in L$ is a root of some non-zero polynomial from $K[X]$.

   (b) Prove that the degree of the minimal polynomial of θ divides $[L:K]$.

4. Let $K=\mathbb{Q}(\sqrt[5]{2})$.

   (a) Prove that K is not normal over $\mathbb{Q}$.

   (b) Find the minimal field extension L of K such that L is normal over $\mathbb{Q}$.

5. (i) Give the definition of a separable field extension. State carefully the criterion for the field $L=K(\alpha)$, where α is a root of an irreducible polynomial $F(X) \in K[X]$, to be separable over K.

   (ii) Give an example of an inseparable field extension of degree 3. (Justify your answer.)

6. Let $L=\mathbb{Q}(\sqrt[4]{2},\sqrt[3]{5})$.

   (a) Find $[L:\mathbb{Q}]$.

   (b) Prove that X^4-2 is the minimal polynomial for $\sqrt[4]{2}$ over $\mathbb{Q}(\sqrt[3]{5})$.

SECTION B

7. Let $\Theta = \sqrt{13 - 5\sqrt{6}}$.

   (a) Find the minimal field extension L of $\mathbb{Q}$ such that $\Theta \in L$ and L is Galois over $\mathbb{Q}$; (Justify your answer.)

   (b) Find the group $Gal(L/\mathbb{Q})$ and all conjugates for Θ over $\mathbb{Q}$.

   (c) Find all subfields K of L such that $[K : \mathbb{Q}] = 2$. (Justify your answer.)

   (d) Apply Galois theory to prove that $\sqrt{2} \notin L$.

8. (i) Find the minimal polynomial for $\alpha = \sqrt[3]{\sqrt{5} - 1}$ over $\mathbb{Q}$.

   (ii) Let $P(T) = T^3 + 2T + 1 \in \mathbb{Q}[T]$.

   (a) Find the Galois group of $P(T)$ over $\mathbb{Q}$.

   (b) Is there $A \in \mathbb{Q}$ such that $\mathbb{Q}(\sqrt[3]{A}) = \mathbb{Q}(\theta)$, where θ is a root of $P(T)$? (Justify your answer.)

   (c) Prove that the splitting field for $P(T)$ over $\mathbb{Q}$ equals $\mathbb{Q}(\theta, \sqrt{-59})$.

9. Let ζ be a 17-th primitive root of unity and $L = \mathbb{Q}(\zeta)$. Prove that:

   (a) there are unique subfields K_1, K_2 and K_3 in L such that $[L : K_3] = 2$, $[L : K_2] = 4$ and $[L : K_1] = 8$;

   (b) $L = K_3(\zeta - \zeta^{-1})$;

   (c) $K_3 = \mathbb{Q}(\zeta + \zeta^{-1})$;

   (d) $K_2 = \mathbb{Q}(\zeta + \zeta^4 + \zeta^{-1} + \zeta^{-4})$.

10. (i) Find all complex roots of the polynomial $T^4 + (9/2)T^2 + 3T + 85/16$.

    (ii) (a) List all subfields of $\mathbb{F}_{5^{12}}$; (Justify your answer.)

    (b) Does this field contain a primitive 13-th root of unity?

    (c) Find an irreducible polynomial $P(X) \in \mathbb{F}_{25}[X]$ such that $\mathbb{F}_{5^{12}} = \mathbb{F}_{25}(\theta)$, where θ is a root of $P(X)$.