Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH3071-WE01

Title: Decision Theory III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

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SECTION A

1. Let $\prec^*$ be a preference ordering on the set $G(\mathcal{R})$ of all gambles over a set of basic rewards $\mathcal{R}$.

    (a) Give the definition of utility function.

    (b) State the existence theorem for utility functions.

    (c) State and briefly explain the Coherence axiom.

2. Let $\mathcal{R}$ be a set of rewards and let $G(\mathcal{R})$ be the set of all gambles over $\mathcal{R}$.

    (a) Define risk aversion for a subject with utility function U on $G(\mathcal{R})$,

    (b) Show that the subject is risk averse if, and only if, her utility function U is concave.

    (c) Assume that the subject's utility function U is twice-differentiable. Write down the subject's local risk aversion r and determine its sign for a risk averse subject.

3. Given two attribute sets $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{R}$ be a reward set such that

$$\mathcal{R} = \mathcal{X} \times \mathcal{Y} = \{(x, y) : x \in \mathcal{X}, y \in \mathcal{Y}\}.$$

   Assume that $\mathcal{X}$ and $\mathcal{Y}$ are mutually independent.

    (a) Let U be a utility function on $G(\mathcal{R})$, and let (x_0, y_0) be such that $U(x_0, y_0) = 0$. Write down the utility function for $(x, y) \in \mathcal{X} \times \mathcal{Y}$ as a function of its marginal utilities and a constant $K \in \mathbb{R}$.

    (b) Let $x_0 \prec^* x_1 \prec^* x_2$, and $y_0 \prec^* y_1 \prec^* y_2$. We define $A = (x_1, y_1)$, $B = (x_1, y_2)$, $C = (x_2, y_1)$, and $D = (x_2, y_2)$. Assume that $A \prec^* B \prec^* D$ and $A \prec^* C \prec^* D$. The subject is offered two gambles: $g_1 = 0.5A \oplus 0.5D$ and $g_2 = 0.5B \oplus 0.5C$. If $g_1 \succ^* g_2$, what can we say about the subject's attitude towards the attributes $\mathcal{X}$ and $\mathcal{Y}$? Specify whether they are complementary, substitutable, or neither, and justify your answer.

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4. 37 voters provide their preference orders over four options, A, B, C and D. The number of voters for specific preference orders are given in the table below.

number of voters:	14	10	8	4	1
1st	A	C	D	B	C
2nd	B	B	C	D	D
3rd	C	D	B	C	B
4th	D	A	A	A	A

To combine their preferences into a group preference, the plurality rule with elimination is used: only the first preference for each voter is considered, and the option which is the first preference of the smallest number of voters is deleted. At the next stage, this is repeated for the remaining options using the given preferences over these options, and so on until one option has more than 50% of the votes. The group preference order results in the logical way, with the winning option ranked first and the one removed first in this process being ranked fourth.

(a) Derive the group preference order over the four options.

(b) Suppose that the 24 voters represented by the first two columns know the votes of all voters. Explain whether or not this would give them an opportunity for tactical voting to their benefit.

(c) Apply the Borda count procedure to combine the voters' preferences as given in the table above. Explain a disadvantage of this procedure and illustrate it using the given preferences.

5. In a particular game, R chooses row 1 or 2, C chooses column 1, 2, 3 or 4. The payoffs to R are as follows

	C1	C2	C3	C4
R1	4	9	6	8
R2	9	3	4	6

The payoff to C is minus the payoff to R.

(a) Use a graphical method to identify the minimax strategies for R and for C, and the value of the game.

(b) Explain why a player might choose to play the minimax strategy.

(c) Explain briefly, using an example, why such games become more complex if the payoff to C is not minus the payoff to R.

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6. Consider a two-person game where the players do not cooperate. Each player chooses independently of the other player one of two options, A or B. The payoffs to the two players are presented in the table below, as $(p1, p2)$ with $p1$ the payoff to Player 1 and $p2$ the payoff to Player 2, if the corresponding options are chosen.

	Player 2	
Player 1	A	B
A	(P, P)	(T, S)
B	(S, T)	(R, R)

Assume that $S < P < R$. For each of the following cases, provide an analysis of this game from the perspective of Player 1, assuming this player uses a minimax strategy. Also comment on important aspects of this game.

(a) $T < S$

(b) $S < T < R$

(c) $T > R$

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SECTION B

7. Detective Inspector Jack Frost opened a cake shop where he sells cakes made using Granny Frost's secret recipe. Three big resellers, "Breaking Bread", "The Rolling Scones", and "Tiers of Joy" have approached Frost with substantial contracts. Frost has three options: (i) work with "Breaking Bread", or (ii) work with "The Rolling Scones", or (iii) work with "Tiers of Joy".

   In order to process orders from these resellers, Frost will have to hire additional staff which will incur additional costs. These costs will have to be entirely recovered from the contract price. There is a risk of not being able to deliver the cakes on time even after hiring extra staff. If that is the case, Frost won't get paid for the order and will make a loss.

   - The staffing costs for fulfilling the "Breaking Bread" order is £50,000. The cost of the ingredients and the manufacturing of the cakes would be £18,000.
   - The staffing costs for fulfilling the order for "The Rolling Scones" is £14,000. The cost of the ingredients and manufacturing of the cakes would be £12,000.
   - The staffing costs for fulfilling the order for "Tiers of Joy" is £55,000. The cost of ingredients and manufacturing would be £24,000.

   The costs related to staffing, ingredients, and manufacturing are fixed for each case regardless of the order size.

   If Frost decides to accept the "Breaking Bread" proposal and completes a large order on time, he will be paid £130,000. The probability of Frost being able to deliver this large order on time is 0.20. He may also choose to take a smaller order for which he would be paid £115,000. The probability of delivering this smaller order would be 0.85 with the same original costs.

   If Frost decides to work with "The Rolling Scones", he may choose to:

   - fulfill a large order of £70,000 with probability of delivery of 0.15; or
   - fulfill a medium-sized order of £65,000 with probability of delivery of 0.80; or
   - fulfill a small order of £60,000 with probability of delivery of 0.95.

   Finally, if Frost decides to help "Tiers of Joy", he may choose to: fulfill a large order of £190,000 with probability of delivery of 0.05; or fulfill a small order of £140,000 with probability of delivery of 0.65.

   Frost can only choose to fulfill one order, and the order has to be fully completed before payment.

   (a) Assuming that Frost wishes to maximize profits, which order should he take?

   (b) Frost is approached by a consultancy firm "The Torte Reform". They ask Frost for £20,000. In return, they will ensure that, if Frost chooses to fulfill the small order for "Tiers of Joy" (£140,000), the order will be delivered on time. Should you accept their offer? Why?

   (c) The current success probability for the largest order (£190,000) is 0.05. By how much would this rate need to be increased such that trying to fulfill this order becomes optimal?

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8. Assume we have a decision problem $[W, D, \phi, L]$ with parameter space W, decision space D, prior distribution ϕ over D, and loss function L given by

$$L(w, d) = |w - d|.$$

(a) Find the Bayes rule for the problem $[W, D, \phi, L]$.

(b) We are now given the problem of estimating the mean parameter w of a normal distribution with known variance 1. The prior probability distribution for w is also a normal distribution with mean 0 and variance 1. Calculate the Bayes rule and the Bayes risk for an immediate decision.

(c) Suppose that we may take a sample of n independent observations from the normal distribution with variance 1 and mean parameter w before estimating w. Show that the posterior distribution for w if we observe $X_i = x_i$, $i = 1, \ldots, n$ is also a normal distribution with mean $\sum_{i=1}^{n} X_i/(n+1)$ and variance $1/(n+1)$.

(d) Find the posterior Bayes rule and Bayes risk when we have observed $X_i = x_i$, $i = 1, \ldots, n$.

Note that for a variable $X \sim N(\mu, \sigma^2)$ (normally distributed with mean μ and variance σ^2), its density function is given by:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-(x-\mu)^2/(2\sigma^2)\right), \ x \in \mathbb{R}.$$

9. (i) Tony and Sue have won a lottery prize and wish to celebrate. They have identified four possibilities, for which their individual utilities are as follows:

	Holiday	New Car	New Kitchen	Big Party
Tony	2	4	8	5
Sue	7	6	3	4

They decide that, if they fail to reach agreement, they will put the prize money in a savings account, for which option Tony has utility 1 and Sue has utility 2.

(a) Sketch the feasible region for this problem, identify the Pareto Boundary and the status quo point.

(b) Find the Nash point and the equitable distribution point for this bargaining problem. What do you recommend Tony and Sue to do?

(c) On reconsidering their utility for saving the money, given economic uncertainties, Tony increases his utility for this option to 6, and Sue increases her utility for saving the money to 3. Explain briefly, without further calculations, whether or not this affects the Nash point and/or the equitable distribution point, and whether such a change of utilities would be to the advantage of Tony, Sue, both or neither.

(ii) The Nash point and the equitable distribution point are each the unique solutions to a certain collection of six axioms for bargaining problems. Five of the axioms are the same in each case, while the sixth axioms differ. Write down the sixth axiom for the Nash point and for the equitable distribution point, and comment briefly on the interpretations of these two axioms.

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10. Suppose that each member of a group of individuals specifies their preference order over a collection of possible rewards. We want to combine the individual preferences into a group preference.

    (a) Explain what is meant by a social welfare function for this problem.

    (b) State Arrow's impossibility theorem, including the collection of four axioms for the group preference relevant to this theorem. Show, by example, that we can find social welfare functions which satisfy any three of these four axioms.

    (c) Give a brief outline proof for Arrow's theorem (sufficient to explain the basic idea of the proof, but not containing the technical details).

    (d) Suppose that each member of the group states utilities for the various rewards. Discuss the problem of combining the individual utilities. (You should state clearly any axioms and theorems which are relevant to this discussion, but you do not need to provide proofs.)

    (e) Explain briefly whether or not Sen's so-called 'Paradox of the Paretian Liberal' is relevant for the method discussed in part (d). Include a description of the axiom of Minimal Liberalism in your explanation.

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