

## **EXAMINATION PAPER**

Examination Session: May

2017

Year:

Exam Code:

MATH3081-WE01

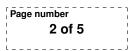
Title:

## Numerical Differential Equations III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use diction	onaries: No	

the best <b>FOUR</b> answers from Section A and the best <b>THREE</b> answers from Section B. Questions in Section B carry <b>TWICE</b> as many marks as those in Section A.	
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Revision:



## SECTION A

- 1. (a) Write down the conditions (in terms of the coefficients  $a_{ij}$ ,  $b_j$  and  $c_j$ ) for a Runge–Kutta scheme to be of order (at least) 3.
  - (b) Find all explicit 2-stage Runge–Kutta schemes of order 2.
- 2. (a) Describe briefly how one obtains the implicit Adams scheme

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + k \sum_{m=0}^{s} B_m \boldsymbol{f}(\boldsymbol{x}^{n-m+1})$$

for some constant coefficients  $B_m$ .

(b) Verify that the case s = 2 gives the scheme

$$x^{n+1} = x^n + \frac{k}{12} [5f(x^{n+1}) + 8f(x^n) - f(x^{n-1})].$$

- 3. (a) State what is meant by *zero stability* for a multistep method.
  - (b) Prove or give a counterexample: every Runge–Kutta method is zero stable.
  - (c) Consider the scheme

$$\boldsymbol{x}^{n+1} - (1+\gamma)\boldsymbol{x}^n + \gamma \boldsymbol{x}^{n-1} = \frac{k}{2} \left[ (3-\gamma)\boldsymbol{f}(\boldsymbol{x}^n) - (\gamma+1)\boldsymbol{f}(\boldsymbol{x}^{n-1}) \right].$$

Work out its order (which may depend on  $\gamma \in \mathbb{R}$ ) and determine what values of  $\gamma$  are suitable for use.

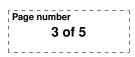
4. Let u = u(x) solve the following boundary-value problem

$$u''(x) - 2u(x) = \sin(x) - 1 \quad \text{for } x \in [0, 1],$$
  
$$u'(0) - u(0) = 0, \qquad u(1) = 0.$$

- (a) Assuming that  $u \in C^3([0, 1])$ , write down the Taylor expansion for u(h) centred at 0 with accuracy  $O(h^3)$ , taking into account that u satisfies the differential equation.
- (b) Based on (a), find a second-order approximation for the boundary condition u'(0) u(0) = 0 in terms of u(0) and u(h). Justify your approximation.
- 5. For the equation  $\partial_t u(x,t) + a \partial_x u(x,t) = 0$  (a is a constant), consider the following approximation scheme

$$\frac{u_m^{n+1} - \frac{1}{2}(u_{m+1}^n + u_{m-1}^n)}{\tau} + a \frac{u_{m+1}^n - u_{m-1}^n}{2h} = 0 \quad \text{with } n \in \mathbb{Z}_+, \ m \in \mathbb{Z}.$$
 (1)

- (a) Considering a particular solution  $u_m^n = \lambda(\phi)^n e^{im\phi}$ , where  $\phi \in [0, 2\pi]$ , to the problem (1), find the amplification factor  $\lambda(\phi)$ .
- (b) Now let  $\tau = rh/a$  for some constant r. Using the spectral stability test, determine under which conditions on the discretisation parameters  $\tau > 0$  and h > 0 the scheme (1) is stable.





6. (a) Consider the iteration

$$oldsymbol{x}^{k+1} = oldsymbol{G}oldsymbol{x}^k + oldsymbol{c}$$

where G is a square matrix, and  $x^{j}$  and c are vectors, with  $x^{0}$  given. Assuming that the equation x = Gx + c has a unique solution, state necessary and sufficient conditions on G that guarantee convergence of the iteration.

(b) Given a real matrix

$$oldsymbol{A} = \left( egin{array}{ccc} lpha & 0 & eta \ 0 & lpha & 0 \ eta & 0 & lpha \end{array} 
ight) \quad ext{with } lpha 
eq 0,$$

and some vector  $b \in \mathbb{R}^3$ , we seek to solve Ax = b using the Gauss–Seidel iteration process. Find the transition matrix G corresponding to A.

(c) Find all values of parameters  $\alpha$  and  $\beta$  such that the Gauss–Seidel iteration process converges for arbitrary initial vector  $\boldsymbol{x}^{0}$ .

## SECTION B

- 7. (a) For a numerical scheme  $\boldsymbol{x}^{n+1} = \Phi(\boldsymbol{x}^n; k)$ , define what is meant by (i) the local truncation error, and (ii) the order of the scheme.
  - (b) For all  $\theta \in [0, 1]$ , determine the order of the  $\theta$ -method,

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + k[\theta \boldsymbol{f}(\boldsymbol{x}^n) + (1-\theta)\boldsymbol{f}(\boldsymbol{x}^{n+1})].$$
(2)

(c) One can solve (2) using the fixed-point iterations  $\boldsymbol{y}_0 = \boldsymbol{x}^n$  and

$$\boldsymbol{y}_{m+1} = \boldsymbol{x}^n + k[\theta \boldsymbol{f}(\boldsymbol{x}^n) + (1-\theta)\boldsymbol{f}(\boldsymbol{y}_m)].$$

Stating any necessary assumptions, prove that these iterations converge.

- (d) Show that if one terminates the fixed-point iterations at  $y_2$ , one obtains a Runge–Kutta scheme; write down its Butcher table.
- 8. (a) Define the notions of stability region and A-stability for one-step schemes.
  - (b) Show that the stability region for the implicit Euler scheme  $\mathbf{x}^n = \mathbf{x}^{n-1} + k\mathbf{f}(\mathbf{x}^n)$  is  $S = \{z \in \mathbb{C} : |z 1| > 1\}.$
  - (c) Let  $\boldsymbol{x} = (x_1, \cdots, x_N)$  and consider the system of equations

$$x'_{j} = -j^{2}x_{j}$$
 for  $j \in \{1, \cdots, N\}$ . (3)

If we are to solve (3) using the implicit Euler scheme, what restrictions (if any) on the timestep k are necessary? Justify your answer.

- (d) Suppose now that we have a fourth-order scheme whose stability region S satisfies  $S \cap \mathbb{R} = (-1, 0)$ . If this scheme were to be used to solve (3), state any necessary restrictions on the timestep k. Justify your answer.
- (e) Show ("from first principles") that, when used to integrate (3), any explicit Runge–Kutta scheme would require a timestep restriction.

9. Let  $\bar{\Omega} = [0,1] \times [0,1]$  and  $u = u(x,y) \in C^1(\bar{\Omega})$  be such that  $u|_{\partial\Omega} = 0$ ; that is u(0,y) = u(1,y) = 0 for  $y \in [0,1]$  and u(x,0) = u(x,1) = 0 for  $x \in [0,1]$ . (a) Prove that

$$|u(x,y)| \leq \int_0^1 |\partial_x u(s_1,y)| \, ds_1, \ \forall (x,y) \in \bar{\Omega},$$
$$|u(x,y)| \leq \int_0^1 |\partial_y u(x,s_2)| \, ds_2, \ \forall (x,y) \in \bar{\Omega}.$$

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Hint: use Newton's formula with respect to x for a fixed y and vice versa. (b) Prove that

$$\int_0^1 \int_0^1 |u(x,y)|^2 \, dx \, dy \le \int_0^1 \int_0^1 |\partial_x u(x,y)| \, dx \, dy \int_0^1 \int_0^1 |\partial_y u(x,y)| \, dx \, dy.$$

Hint: estimate  $|u(x, y)|^2$ , combining inequalities from (a).

Now let  $u_{m,n}, m \in \overline{0, M}, n \in \overline{0, N}$  be a grid function such that

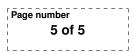
$$u_{0,n} = u_{M,n} = 0, \ n \in \overline{0, N}, \qquad u_{m,0} = u_{m,N} = 0, \ m \in \overline{0, M}.$$

(c) Prove the discrete analogue of (a), namely

$$\begin{aligned} |u_{m,n}| &\leq \sum_{k=1}^{M} |u_{k,n} - u_{k-1,n}|, \ \forall (m,n), \text{ where } m \in \overline{0,M}, \ n \in \overline{0,N}; \\ |u_{m,n}| &\leq \sum_{k=1}^{N} |u_{m,k} - u_{m,k-1}|, \ \forall (m,n), \text{ where } m \in \overline{0,M}, \ n \in \overline{0,N}. \end{aligned}$$

(d) Prove the discrete analogue of (b), namely

$$\sum_{m=0}^{M} \sum_{n=0}^{N} |u_{m,n}|^2 \le \left(\sum_{n=0}^{N} \sum_{k=1}^{M} |u_{k,n} - u_{k-1,n}|\right) \left(\sum_{m=0}^{M} \sum_{k=1}^{N} |u_{m,k} - u_{m,k-1}|\right).$$





10. Consider the eigenvalue problem for the following difference scheme

$$\frac{u_{k+1} - u_{k-1}}{2h} = -\lambda u_k \quad \text{for } 1 \le k \le N - 1 \quad \text{with } hN = 1, \tag{4}$$

$$u_0 = 0, \quad u_N = 0.$$
 (5)

- (a) Find the roots  $\mu_1$  and  $\mu_2$  of the characteristic polynomial associated to the difference equation (4). Find the values of  $\mu_1 + \mu_2$  and  $\mu_1 \mu_2$ .
- (b) Under the assumption  $\mu_1 = \mu_2 := \mu$ , the general solution to the difference equation (4) takes the form

$$u_k = c_1 \mu^k + c_2 k \mu^k.$$

Taking into account the boundary conditions (5), find all  $\lambda \in \mathbb{C}$  such that the problem (4)–(5) has a nonzero solution.

(c) Under the assumption  $\mu_1 \neq \mu_2$ , the general solution to the difference scheme takes the form

$$u_k = c_1 \mu_1^k + c_2 \mu_2^k.$$

Taking into account the boundary conditions (5), find all  $\lambda \in \mathbb{C}$  such that problem (4)–(5) has a nonzero solution.