EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH3091-WE01

Title: Dynamical Systems III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. A linear n-dimensional dynamical system obeys the differential equations $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, where A is a constant $n \times n$ matrix and $\boldsymbol{x}$ an n-dimensional state vector.

   (a) Derive the ordinary differential equation that the time evolution map obeys and relate its solution to the exponential map.

   (b) Determine how the exponential $\exp(A)$ of the matrix A transforms under a similarity transformation $A' = M^{-1}AM$.

   (c) Use the answer of part (b) to determine the exponential $\exp(A)$ of the matrix

$$A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} .$$

2. The differential equation for the pendulum is $\ddot{x} + \sin x = 0$.

   (a) Write this equation as a two-dimensional dynamical system with phase variables (x, y), where $y = \dot{x}$.

   (b) Show that the quantity

$$V(x, y) = 1 - \cos x + \frac{1}{2}y^2$$

   is a first integral of the dynamical system, namely it is a conserved quantity along trajectories. Is the origin of the phase space Liapounov stable?

   (c) Consider the pendulum with friction

$$\ddot{x} + a\dot{x} + \sin x = 0 \ , \quad a > 0 \text{ is a constant} .$$

   Show that the function V of part (b) is no longer a first integral, but it is a Liapounov function. What is your physical intuition about the stability properties of this system near the origin? Is the origin asymptotically stable? Explain your answer. Demonstrate your expectations with a linearized analysis around the origin.

3. Consider the six-dimensional linear dynamical system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ with

$$A = \begin{pmatrix} -5 & -1 & 0 & 0 & 0 & 0 \\ 1 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 4 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 \end{pmatrix} .$$

(a) Find in coordinate form:

(i) the stable manifold M_-,

(ii) the unstable manifold M_+,

(iii) the center manifold M_0.

[Hint: notice that by relabelling the variables x^i you can rearrange the matrix A to bring A into a more convenient form.]

(b) Now consider a non-linear deformation of a linear system with general constant matrix A

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{G}(\boldsymbol{x}) .$$

$\boldsymbol{G}(\boldsymbol{x})$ is a continuously differentiable function that vanishes and has zero first derivatives at the origin. When is the origin a hyperbolic fixed point?

(c) If the origin is a hyperbolic fixed point for a system like that in part (b), and the unstable manifold M_+ of its linearized approximation is non-empty, can you employ the stable manifold theorem to argue that the origin is not Liapounov stable?

4. Consider a one-dimensional dynamical system $\dot{x} = f(x, \mu)$ with $f(x, \mu)$ infinitely differentiable in both its arguments.

(a) State necessary conditions for the system to undergo a local bifurcation at a point (x_*, μ_*).

(b) Suppose the system undergoes a pitchfork bifurcation at $(x, \mu) = (0, 0)$. Give sufficient conditions for $f(x, \mu)$ for this to happen, and state the corresponding normal form. Would a pitchfork bifurcation be robust under small perturbations?

Consider now $f(x, \mu) = (x^2 - \mu^2)(x^2 - (\mu + 2)^2)$.

(c) Work out the conditions under part (a) and find all bifurcation points.

(d) Draw a bifurcation diagram. State the type of all the bifurcation points.

5. Consider the three-dimensional dynamical system of the form

$$\dot{x} = -y \qquad \dot{y} = x \qquad \dot{z} = -x^2 - y^2$$

(a) Determine the fixed points of the system.

(b) Why is the system not Hamiltonian?

(c) State the definitions of an omega limit point and an omega limit set.

(d) Determine the omega limit set of every point in $\mathbf{R}^3$ for the given dynamical system. Briefly justify your answer.

(e) Let $B = \{(x, y, z) \mid x^2 + y^2 + z^2 < 1\}$ be the ball with unit radius and $\phi(t, B)$ be the domain obtained by evolving all points in B for some time t according to the above dynamical system. Compute $\mathrm{Vol}(\phi(t, B))$, the volume of $\phi(t, B)$.

6. Consider a C^1 planar dynamical system

$$\dot{x} = f(x, y) \qquad\qquad \dot{y} = g(x, y)$$

Suppose it has a compact positively invariant set B homeomorphic to a disk.

(a) Give the definition of the index $I(\gamma)$ for a closed curve γ in terms of $f(x, y)$ and $g(x, y)$.

(b) Now let γ be the boundary of B. What is $I(\gamma)$? Use a sketch to explain your answer.

(c) Suppose the system has exactly two fixed points in B. Explain why they cannot both be hyperbolic.

Now suppose that

$$f(x, y) = -x + \alpha y \qquad g(x, y) = -y$$

(d) For which values of α is the disk $B = \{(x, y) \mid x^2 + y^2 \leq 1\}$ positively invariant? Explain your answer.

SECTION B

7. Consider the two-dimensional non-linear dynamical system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + (x_1 - x_2)^3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{with } A = \begin{pmatrix} -5 & 3 \\ -6 & 4 \end{pmatrix} .$$

   (a) Perform a linear transformation $\boldsymbol{x} = M\boldsymbol{y}$ that diagonalizes the matrix A. Determine M.

   (b) Is the origin a hyperbolic fixed point?

   (c) What is the form of the non-linear equations of the dynamical system in the $\boldsymbol{y}$ coordinates?

   (d) Solve the non-linear equations in the $\boldsymbol{y}$ coordinates. Then, transform the solution back to the $\boldsymbol{x}$ coordinates.

   (e) Sketch the phase flow in the $\boldsymbol{x}$ coordinates and determine the stable and unstable manifolds. Is the stable manifold theorem obeyed?

8. Consider a second-order ordinary differential equation (ODE) of the general form

$$\ddot{x} = g(\dot{x}) + h(x)$$

where the functions g, h are assumed to be C^1 functions with the following properties:

$$g(0) = 0 \ , \quad \left.\frac{dg}{dx}\right|_{x=0} = a > 0 \ ,$$
$$h(x) = 0 \ \Leftrightarrow \ x = \pm 1 \ , \quad \left.\frac{dh}{dx}\right|_{x=\pm 1} = \pm b \in \mathbb{R} \ .$$

   (a) Put the above ODE in canonical form by setting $\dot{x} = y$.

   (b) Determine all the critical points of the resulting dynamical system and the constant matrix A that appears in the linear equations

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

   around each critical point.

   (c) Does the system undergo any qualitative changes (bifurcation) as the parameter b is varied at fixed a? How are such changes manifested in the matrix A around each fixed point? Determine the different behaviours of the system (by chacterizing each fixed point as a source, sink, focus etc.). Explicitly mention in what range of the parameters a, b each of these different behaviours occurs.

   (d) Focus on the regime of parameters where $b > \frac{a^2}{4}$. Solve the linear equations explicitly around each fixed point and draw the local phase diagrams.

   (e) Join the local trajectories that you found in part (d) and sketch a possible global structure of the phase diagram.

9. Consider a planar dynamical system of the form

$$\dot{x} = x - (1 + \theta(x))x^3 - y \qquad \dot{y} = y - 3x^2 y + x$$

where $\theta(x)$ is the step function which is defined as follows: $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$.

   (a) Given an initial point (x_0, y_0), does the dynamical system have a solution? Is it unique? Briefly explain your answer.

   (b) State and prove Bendixson's criterion.

   (c) Show that the above system cannot have periodic orbits that lie entirely in the interior of the region $a < x < b$ for some $a < 0$ and some $b > 0$. Determine the largest possible magnitudes for a and b that you can obtain from Bendixson's criterion.

   (d) State the Poincaré-Bendixson theorem.

   (e) Is the fixed point at the origin of the above dynamical system asymptotically stable? Give a proof of your answers by appealing to one or more known theorems.

   (f) You are given that the system has an absorbing set and no other fixed points. What can you deduce about the nature of the omega limit set of every point besides the origin? Give a proof of your answer by appealing to one or more known theorems.

10. (a) Define what it means for two dynamical systems to be topologically conjugate.

Using a map $\boldsymbol{h}(x, y) = (h_x(x, y), h_y(x, y)) = (x, y + p(x))$ one may show that the following two systems are topologically conjugate on $\mathbf{R}^n$:

$$\begin{aligned} \text{System I:} \quad & (\dot{x}, \dot{y}) = \big(x(x+1), -y\big) \\ \text{System II:} \quad & (\dot{x}, \dot{y}) = \big(x(x+1), -y + 4x^3 - 12x + 3\big) \end{aligned}$$

   (b) Derive a differential equation for $p(x)$. Show that it can be solved by a second-degree polynomial.

   (c) What is the inverse map $\boldsymbol{h}^{-1}(x, y)$?

   (d) Suppose $V(x, y)$ is a first integral for system II. Give a first integral in terms of $V(x)$ for system I, and prove your answer.

   (e) Out of the three one-dimensional dynamical systems $\dot{x} = x$ and $\dot{x} = 2x^2$ and $\dot{x} = 3x^3$ there are two that are topologically conjugate to each other on some symmetric interval I around the origin. Prove this by determining a homeomorphism and give the associated maximal extent of I. Explain also why the remaining system is not topologically conjugate to any of the other two in any domain around the origin.

   (f) Suppose a dynamical system has a compact positively invariant set D. For a point $p \in D$, state four properties of its omega limit set $\omega(p)$.