EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH3111-WE01

Title:
Quantum Mechanics III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. Consider a two dimensional Hilbert space spanned by the orthonormal basis $S = \{|1\rangle, |2\rangle\}$. The action of the physical observable $\hat{\Omega}$ on S is given by

$$\hat{\Omega}\,|1\rangle = |1\rangle + \sqrt{3}\ |2\rangle$$
$$\hat{\Omega}\,|2\rangle = b\ |1\rangle - |2\rangle\,,$$

with b a complex constant.

(a) Find the matrix form of $\hat{\Omega}$ in the basis S and fix the constant b.

(b) What are the possible outcomes in a measurement of $\hat{\Omega}$?

(c) A measurement of $\hat{\Omega}$ yields the maximum possible outcome. Write the state right after the measurement as a linear combination of the basis vectors S.

2. Consider a quantum mechanical particle of mass m in two dimensions subject to a potential V which classically takes the form

$$V(x, y) = c_x^2\, x^2 + c_y^2\, y^2$$

for some real constants c_x and c_y.

(a) Write down the quantum mechanical Hamiltonian $\hat{H}$ of the system.

(b) Compute the commutator of $\hat{H}$ with the angular momentum operator $\hat{L}$.
Hint: You can use $\hat{L} = \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x$.

(c) Find the condition that c_x and c_y have to satisfy for the expectation value of $\hat{L}$ to be conserved in time.

3. The Fermionic simple harmonic oscillator is based on the operator $\hat{b}$ which satisfies

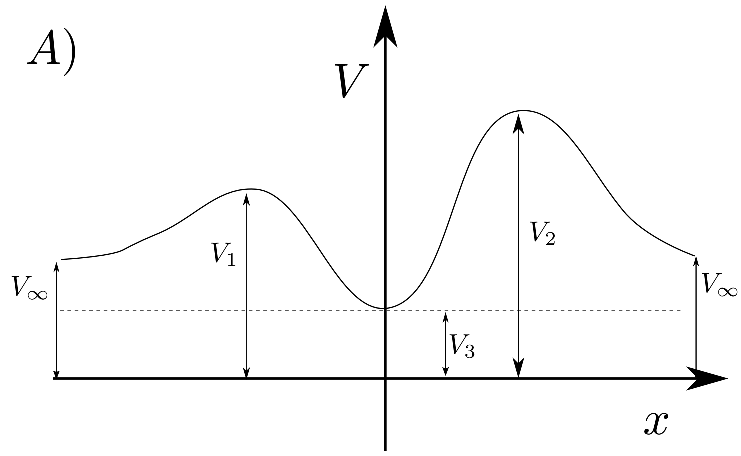
$$\hat{b}^2 = 0, \qquad \{\hat{b}^\dagger, \hat{b}\} \equiv \hat{b}^\dagger\,\hat{b} + \hat{b}\,\hat{b}^\dagger = 1$$

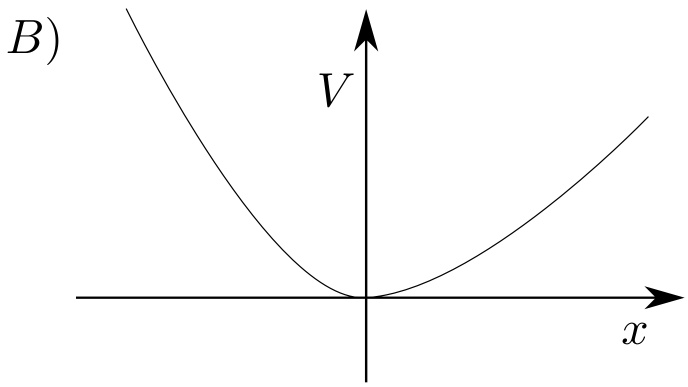
(a) What is the operator $(\hat{b}^\dagger)^2$ equal to?

(b) Consider the operator $\hat{N} = \hat{b}^\dagger\,\hat{b}$. Compute $\hat{N}^2$ and find the constant a which satisfies $\hat{N}^2 = a\,\hat{N}$.

(c) Find all the eigenvalues of the operator $\hat{N}$. Justify your answer.

4. The potentials for two different one-dimensional quantum systems are shown in the figure below.





A quantum particle is propagating in one dimension under the influence of these potentials. Answer the following questions:

(a) Is the energy eigenspectrum of the particle in the potential A) discrete or not? If only part of the spectrum is discrete, explain for which values of the energies this happens.

(b) Is the energy eigenspectrum of the particle in the potential B) discrete or not? If only part of the spectrum is discrete, explain for which values of energies this happens.

(c) For the potential B) state for which values of the variable x the wave functions are oscillatory and for which values of x they are not. If the range in which this happens depends on the energy E of the particle, please clearly say at which values of x the wave function changes.

(d) For the potential B), in the region where the wave function is not oscillatory, sketch how the probability density behaves as a function of x. Where the wave function is oscillatory, indicate in which direction the amplitude of oscillations increases for highly excited (semi-classical) energy eigenstates. Around which point is the probability to find the semiclassical particle the largest?

5. A quantum particle of energy $E \gg V_0$ and mass m is placed inside a potential well with a sloped bottom and infinitely high walls; the potential $V(x)$ is given by

$$V(x) = \begin{cases} \infty & x \le 0 \quad \text{and} \quad x \ge L\,, \\ \dfrac{V_0}{L} x & 0 < x < L\,, \end{cases}$$

where $L, V_0 > 0$.

(a) Apply the WKB approximation to this system and derive the form of the wave function. Explain why it is justified to use the WKB approximation for this system.

(b) Using the wave function derived above, derive the equation for the energies that this particle can have. You do not need to solve this equation explicitly.

6. A particle is propagating in a potential given by

$$V(r, \theta, \varphi) = \alpha r^2\,,$$

where $\alpha > 0$ and r, θ, φ are spherical coordinates. The Laplacian in spherical coordinates is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{1}{r^2}\hat{L}^2\,,$$

where $\hat{L}^2$ is the square of the angular momentum operator in the coordinate representation.

(a) By separation of variables in spherical coordinates, write down the angular equation for this system. Without explicitly solving this equation, state which quantum numbers characterise the solutions of this equation.

(b) By separation of variables in spherical coordinates, write down the radial equation for this system.

(c) Evaluate the energy of the ground state, if its wave function is given by

$$R(r) = B e^{-\frac{1}{2\hbar}\sqrt{2m\alpha}\, r^2}$$

Here B is a normalisation constant which is irrelevant.

(d) Is there an alternative way of determining the ground state energy of this system, without knowing the wave function given in the previous part?

SECTION B

7. The quantum mechanical Hamiltonian describing a particle with spin $1/2$ in a magnetic field along the z direction of amplitude B is given by

$$\hat{H} = \frac{\mu B}{2} \hat{\sigma}_z \,.$$

We will be ignoring the particle's orbital motion and the vector space of states will be two dimensional. The given constant μ is the particle's magnetic moment and the Pauli matrices are given by

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Find the energy eigenstates and energy eigenvalues.

(b) At $t = 0$, we perform a measurement of $\hat{\sigma}_x$ which yields the maximum value possible. Find the state right after the measurement.

(c) Write the particle's state after a time interval $t > 0$.

(d) Compute the expectation values $\langle\, \hat{\sigma}_x \,\rangle$, $\langle\, \hat{\sigma}_y \,\rangle$ and $\langle\, \hat{\sigma}_z \,\rangle$ at a later time $t > 0$.

8. Consider the isotropic two dimensional simple harmonic oscillator of frequency ω and mass m. The quantum mechanical Hamiltonian $\hat{H}$ and angular momentum $\hat{L}$ operators can be expressed as

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1\right)$$
$$\hat{L} = \hbar \left(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}\right)$$

with the only non-trivial commutators among the operators $\hat{a}^\dagger$, $\hat{a}$, $\hat{b}^\dagger$ and $\hat{b}$ being

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{b}, \hat{b}^\dagger] = 1\,.$$

We define the unit norm states

$$|\,m, n\rangle = \frac{1}{\sqrt{m!}}\frac{1}{\sqrt{n!}}\,(\hat{a}^\dagger)^m\,(\hat{b}^\dagger)^n\;|\,0,0\rangle$$

for m and n non-negative integers and with $|\,0,0\rangle$ being the correctly normalised ground state satisfying $\hat{a}\,|\,0,0\rangle = \hat{b}\,|\,0,0\rangle = 0$.

(a) Use induction to show that $[\hat{a}, (\hat{a}^\dagger)^m] = m\,(\hat{a}^\dagger)^{m-1}$ and $[\hat{b}, (\hat{b}^\dagger)^n] = n\,(\hat{b}^\dagger)^{n-1}$ with m and n being positive integers.

(b) Compute the commutator between $\hat{H}$ and $\hat{L}$. Is angular momentum conserved? Is the Hamiltonian invariant under small rotations? Justify your answers.

(c) Show that the states $|\,m, n\rangle$ are simultaneous eigenstates of $\hat{H}$ and $\hat{L}$. Give the expressions for the corresponding eigenvalues in terms of m and n.

(d) Use the results of the previous parts of this question to deduce the inner product $\langle\, m_1, n_1\,|\,m_2, n_2\,\rangle$ without any computation.

(e) At $t = 0$ the system is in the state $|\,\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\,(|\,1,0\rangle + |\,0,0\rangle)$. What is the expectation value of $\langle\,\hat{L}\,\rangle$ at a later time $t > 0$?

9. The simple harmonic oscillator (SHO) in one dimension has the Hamiltonian

$$\hat{H}_0 = \frac{1}{2}\kappa^2 x^2$$

and is perturbed by the Hamiltonian

$$\hat{H}' = \alpha x^3 + \beta x^4 \,,$$

where $\alpha, \beta > 0$ are two small constant parameters, which are of the same order.

(a) If you would want to solve this system in perturbation theory, would you need to use non-degenerate or degenerate perturbation theory? Motivate your answer.

(b) Derive the general expression for the first order correction to the energy in perturbation theory applicable to this system.

(c) Evaluate the expression for the energy correction which you have obtained above explicitly for the n-th energy eigenstate of the SHO with perturbation $\hat{H}'$ given above.

(d) Derive the general expressions for the first order correction to the wave function in perturbation theory applicable to this system.

Hint: When doing this problem you may need the following relations between the $\hat{x}, \hat{p}$ and creation/annihilation operators $\hat{a}^\dagger, \hat{a}$ of the SHO,

$$\hat{a} = \sqrt{m/2\hbar\omega}(\omega\hat{x} + im\hat{p})\,,$$
$$\hat{a}^\dagger = \sqrt{m/2\hbar\omega}(\omega\hat{x} - im\hat{p})\,,$$

so that $[\hat{a}, \hat{a}^\dagger] = 1$.

10. A quantum particle of mass m has a Hamiltonian given by

$$\hat{H} = \hat{H}_0 + \alpha\hat{r}^4 \quad \text{with} \quad \hat{H}_0 = \frac{1}{2}m\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right),$$

where $\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$. You may assume without proof that $[\hat{L}^2, \hat{H}_0] = [\hat{L}_z, \hat{H}_0] = 0$ where $\hat{L}^2$ is the square of the angular momentum operator and $\hat{L}_i$ $(i = x, y, z)$ are its components.

(a) Prove that

$$[\hat{L}^2, \hat{r}^4] = 0 \quad \text{and} \quad [\hat{L}_z, \hat{r}^4] = 0\,. \tag{1}$$

When proving these expressions you are not allowed to go into the coordinate representation.

(b) What are the implications of (1) on the energy eigenstates of the Hamiltonian $\hat{H}$? What are the quantum numbers which label the energy eigenstates of the Hamiltonian $\hat{H}$?

(c) Let $|lm\rangle$ be a unit-normalised eigenstate of the operators $\hat{L}^2$ and $\hat{L}_z$. Evaluate the following expectation values: $\langle\hat{L}_i^2\rangle \equiv \langle lm|\hat{L}_i^2|lm\rangle$ for $i = x, y, z$ and $\langle\hat{L}^2\rangle \equiv \langle lm|\hat{L}^2|lm\rangle$.

(d) Show that $\langle\hat{L}^2\rangle = \langle\hat{L}_x^2\rangle + \langle\hat{L}_y^2\rangle + \langle\hat{L}_z^2\rangle$ as expected.