Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH3141-WE01

Title: Operations Research III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. (a) Solve the following linear program using the simplex method:

$$\begin{aligned} \max\ & 4x_1 + 3x_2 \\ \text{subject to}\ & 2x_1 + 2x_2 \le 3 \\ & 3x_1 + 2x_2 \le 4 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

   (b) Write down the dual of the above linear programming problem.

   (c) Identify the dual solution directly from the final simplex table that you found earlier.

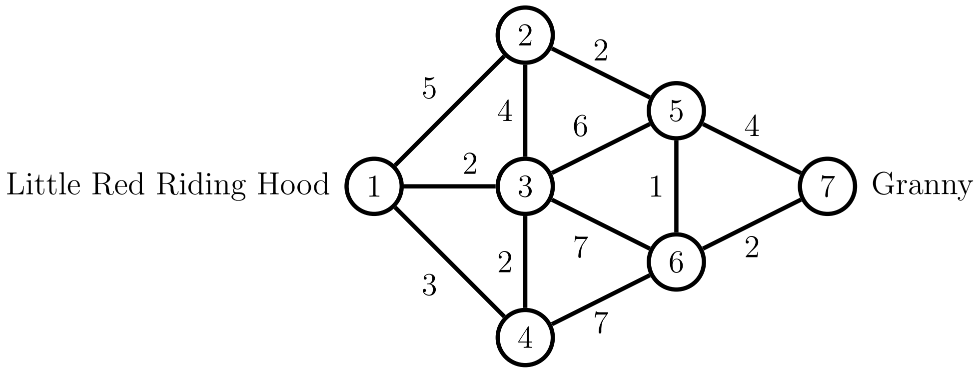
2. Consider the transportation problem with costs, supplies, and demands respectively given by:

$$[c_{ij}] = \begin{bmatrix} 2 & 3 & 1 & 5 \\ 3 & 1 & 4 & 2 \end{bmatrix} \qquad [a_i] = \begin{bmatrix} 40 \\ 60 \end{bmatrix} \qquad [b_j] = \begin{bmatrix} 20 & 25 & 25 & 30 \end{bmatrix}$$

   (a) Find the optimal transportation scheme.

   (b) Assume c_{23} is decreased from 4 to 1. Derive the new optimal transportation scheme.

3. (a) Little Red Riding Hood wants to visit Granny. The following routes are available to her, where the numbers on the arcs denote distances between nodes:



   Apply Dijkstra's algorithm to identify all shortest paths from Little Red Riding Hood (node 1) to Granny (node 7).

   (b) The Woodcutter warns Little Red Riding Hood that the Big Bad Wolf has set an ambush on the arc between nodes 5 and 6. Use Dijkstra's algorithm to identify all shortest paths from Little Red Riding Hood (node 1) to Granny (node 7) not containing this arc.

4. An investor has £50,000 to invest in three different opportunities. At some point in the future these will supply a return to the investor with net present value (in units of £10,000) of $R_j(x)$ when $x > 0$ is invested as follows:

$$R_1(x) = 3x + 7, \quad R_2(x) = 5x + 1, \quad R_3(x) = 4x + 4$$

with $R_1(0) = R_2(0) = R_3(0) = 0$. The minimum investment unit is £10,000. Use dynamic programming to find the allocation of the 5 units to achieve the return with maximal net present value.

5. (a) Derive the Economic Order Quantity formula for inventory control where no shortages are allowed. State carefully any necessary assumptions and define any notation used.

   (b) A computer store sells on average 30 computers per month. The cost of holding a computer in inventory for a month is 2 pounds. It costs 150 pounds each time the store places an order and the cost price per computer is 50 pounds.

   Find an optimal order size and optimal length of a cycle. Explain, without proof, how to proceed if the optimal solutions are not integers.

6. The successive states of a system form a Markov Chain on k states with equilibrium probabilities $\pi = (\pi_1, ..., \pi_k)$. Let $C(j)$ be the cost for the system to be in state j.

   (a) State and derive the formula for the long-run average (LRA) expected cost per unit time.

   (b) Let $k = 3$. Find the LRA for a system with the vector of costs $C = (-1, 5, 2)$ and the transition matrix

$$P = \begin{pmatrix} 0 & 0.1 & 0.9 \\ 0.3 & 0.1 & 0.6 \\ 0.6 & 0.4 & 0 \end{pmatrix}.$$

SECTION B

7. Consider a case where someone has solved the following linear programming problem:

$$\max\ 5x_1+4x_2+3x_3+3x_4 \tag{1a}$$
$$\text{subject to } 3x_1+4x_2+x_3+2x_4 \le 3 \tag{1b}$$
$$2x_1+x_2+3x_3+4x_4 \le 5 \tag{1c}$$
$$x_1+3x_2+3x_3+2x_4 \le 2 \tag{1d}$$

and subject to all $x_i \ge 0$. The initial simplex table for this problem is

T_0	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
z	-5	-4	-3	-3	0	0	0	0
s_1	3	4	1	2	1	0	0	3
s_2	2	1	3	4	0	1	0	5
s_3	1	3	3	2	0	0	1	2

(2)

and the final simplex table is

T_*	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
z	0	$7/2$	0	1	$3/2$	0	$1/2$	$11/2$
x_1	1	$9/8$	0	$1/2$	$3/8$	0	$-1/8$	$7/8$
s_2	0	$-25/8$	0	$3/2$	$-3/8$	1	$-7/8$	$17/8$
x_3	0	$5/8$	1	$1/2$	$-1/8$	0	$3/8$	$3/8$

(3)

Now consider a similar linear programming problem, where the constraints are the same, but where the objective function is modified to

$$\max 5x_1 + 4x_2 + (3-\epsilon)x_3 + (3+\epsilon)x_4 \tag{4}$$

where ϵ is an arbitrary value in $\mathbb{R}$. For this modified problem:

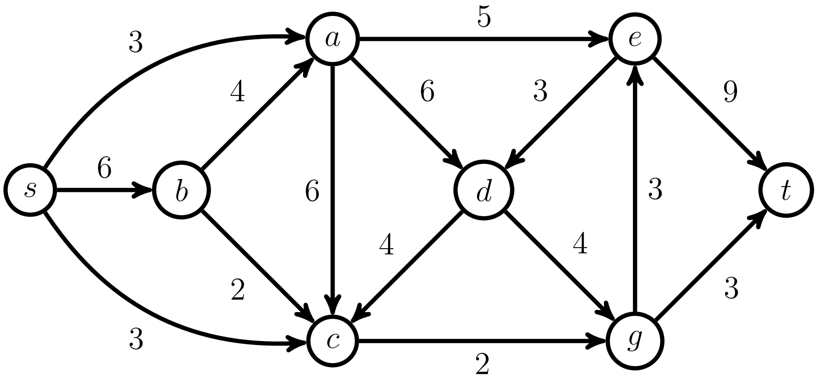
(a) For what values of $\epsilon \in \mathbb{R}$ does the basis $\{x_1, s_2, x_3\}$ give a basic feasible solution?

(b) For what values of $\epsilon \in \mathbb{R}$ does the basis $\{x_1, s_2, x_3\}$ give an optimal basic feasible solution?

(c) In case the basis $\{x_1, s_2, x_3\}$ gives an optimal basic feasible solution for the modified problem, for what values of $\epsilon \in \mathbb{R}$ is the optimal solution unique?

(d) Suggest a pivot to move from the basis $\{x_1, s_2, x_3\}$ to an improved basis, for the case $\epsilon = 8$ (you do not need to perform the pivot).

8. Consider a city whose road network can be represented by the following flow network, with capacities as labelled:



We are specifically interested in the flow from s to t.

(a) Apply the Ford-Fulkerson algorithm to find the maximum flow from the source s to the terminus t.

(b) Find a cut separating s and t with minimal capacity.

(c) City planners are interested in increasing the maximal flow from s to t by increasing the capacity of just one of the arcs. Based on your previous working, identify a single arc that has a good opportunity for increasing the maximal flow, and provide a theoretical argument based on duality as to why your arc is a good one. Increase the capacity of this arc by 3 and find the new maximal flow, and a new cut between s and t with minimal capacity.

9. A machine is inspected at the start of daily production and it can be in one of three states, *good, fair* or *bad* denoted by 1, 2, or 3, respectively. At the end of production each day, the machine can be *repaired* at cost R which brings it to the good state next morning. If it is not repaired, its state changes from i to j with chance p_{ij} where $p_{ij} = 0$ if $j < i$, i.e. the state of the machine cannot improve unless it is repaired. The proportion of production which is defective is ρ_i when the machine starts the day in state i, and the immediate cost of this is $\rho_i C$. Let X_t denote the state of the machine at the beginning of day t.

   (a) Find an expression for the probability that the machine is in the bad state after two days' production given that it starts in the good state and is not repaired.

   (b) Suppose from now on that $p_{11} = 0.6$, $p_{12} = 0.3$, $p_{13} = 0.1$, $p_{22} = 0.7$, and $p_{23} = 0.3$. Consider the policy where the machine is repaired only when it is in the bad state. Write down the one-step transition probability matrix for the Markov chain $X = \{X_t\}$ under this policy and find its stationary distribution. Write down the immediate cost for each state-action pair and find an expression for the long-run average expected cost of this policy in terms of R and $\rho_i C$.

   (c) Let the parameter values be: $R = 1$, $C = 2$, $\rho_1 = 0$, $\rho_2 = 0.2$, and $\rho_3 = 0.3$, and let the discount factor be $\alpha = 0.9$. With the transition probabilities defined above, find the solution to the one-step equations for the total discounted expected cost using the policy S of only repairing the machine when it enters the bad state. Using this solution, run one iteration of the policy improvement algorithm starting from policy S.

10. (i) An optimization problem can be divided into N stages with possible states $s \in \mathcal{S}_n$ and actions $x \in \mathcal{A}_n$ at stage $n \leq N$. Let $\mathcal{S}_{N+1}$ denote the set of final states. The probability of going to state $j \in \mathcal{S}_{n+1}$ from state i after action x in stage n is $p_n(j \mid i, x)$. The immediate return from taking action x in state i at stage n and jumping to state j at stage $n+1$ is $R_n(i, j; x)$ and the objective is to maximize the *expected total return*.

What is a *policy*? Explain the probabilistic dynamic programming method for finding the optimal policy.

(ii) Three investment schemes (A, B, and C) accept investments in multiples of £3000.

Schemes A and B mature after one year, and generate returns distributed as follows:

	annual return on £3000 investment	probability
A	2000	0.2
	5000	0.8
B	0	0.1
	4000	0.4
	6000	0.5

Scheme C matures after two years, generating a guaranteed return as follows:

	two-year return on £3000 investment	probability
C	7000	1.0

You have £4000 to invest at the start of a period of N years.

At the end of a year that does not bring the investment period to a close, any return from that year, together with any amount not invested that year, is available to invest for the next year. Investment may be split between the schemes as desired (provided the rule that investments must be in units of £3000 is respected).

Use stochastic dynamic programming to find the optimal (in terms of expected total profit) investment strategy for the case where

(a) $N = 1$;

(b) $N = 2$.