

EXAMINATION PAPER

Examination Session: May

2017

Year:

Exam Code:

MATH3171-WE01

Title:

Mathematical Biology III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dicti	onaries: No	

Instructions to Candidates:	Credit will be given for: the best FOUR answers from Section and the best THREE answers from S Questions in Section B carry TWICE in Section A.	Section B.	arks as those
-----------------------------	--	------------	---------------

Revision:

SECTION A

1. Fourier transform and Green's method Consider the following P.D.E. and its fundamental problem:

$$\frac{\partial^2 c_f}{\partial t^2} = D \frac{\partial^2 c_f}{\partial x^2}, \quad c_f(x,0) = \delta(x), \quad \frac{\partial c_f}{\partial t}(x,0) = 0, \quad c_f(\pm \infty, t) = 0.$$
(1)

Here $\delta(x)$ is the Dirac delta distribution and D > 0. We define the Fourier Transform as

$$F[c](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} c(x) dx,$$

and its inverse as

$$F^{-1}[\hat{c}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \hat{c}(k) \mathrm{d}k.$$

(a) Show that (1) implies

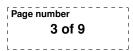
$$F[c_f](k) = \frac{1}{\sqrt{2\pi}}\cos(k\sqrt{D}t).$$

(b) Using Green's method solve the problem

$$\begin{aligned} \frac{\partial^2 c}{\partial t^2} &= D \frac{\partial^2 c}{\partial x^2}, \quad c(x,0) = g(x), \quad \frac{\partial c}{\partial t}(x,0) = 0, \quad c(\pm \infty, t) = 0, \\ g(x) &= \begin{cases} 0 & x < -a, \\ 1 & -a \le x \le a, \\ 0 & x > a \end{cases} \end{aligned}$$

and describe briefly the changing shape of the distribution over time. You may use the result

$$F[\cos(ax)](k) = \sqrt{\frac{\pi}{2}} \left(\delta(k-a) + \delta(k+a)\right).$$



2. P.D.E. stability Consider the following P.D.E. and boundary conditions

$$\frac{\partial^2 y}{\partial x^2} + (1-y)\frac{\partial y}{\partial x} + \alpha y^3 = \frac{\partial y}{\partial t}, \quad y(0,t) = 0, \quad y(L,t) = 0,$$

where $\alpha > 0$ is a constant. There is a homogeneous equilibrium y = 0.

- (a) Assess the stability of this equilibrium.
- (b) How would this conclusion have changed if the problem had instead been

$$\frac{\partial^2 y}{\partial x^2} + (1-y)\frac{\partial y}{\partial x} + \alpha y = \frac{\partial y}{\partial t}, \quad y(0,t) = 0, \quad y(L,t) = 0?$$

State any conditions on domain length L, dependent on the value of the parameter α , which will ensure equilibrium y = 0 is stable.

3. Limit Cycles and the Hopf-bifurcation Consider the following activator-inhibitor system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a - bu + \frac{u^2}{v},$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = u^2 - v,$$

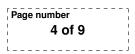
where a, b > 0 are **real** constants.

- (a) Find the single equilibrium of the system.
- (b) Show that this equilibrium exhibits a Hopf-bifurcation when

$$b = \frac{a+1}{1-a},$$

and $a \neq 1$.

(c) Find a quadratic equation for b which gives the boundary between oscillatory and non-oscillatory behaviour of the linearised system. To simplify your algebra make the substitution $\gamma = \frac{2}{a+1} - 1$.



4. Turing analysis Consider the following non-dimensionalised reaction-diffusion system for scalar densities u and v

$$\frac{\partial u}{\partial t} = \nabla^2 u + \gamma F(u, v),$$

$$\frac{\partial v}{\partial t} = D\nabla^2 v + \gamma G(u, v),$$
(2)

Exam code

MATH3171-WE01

where D > 0 and $\gamma > 0$ are constants. We assume **no-flux** boundary conditions on a Cartesian domain $[0, L_1] \times [0, L_2]$ with coordinates (x_1, x_2) . The Turing conditions for pattern formation are

$$F_u + G_v < 0, \quad F_u G_v - G_u F_v > 0, \tag{3}$$

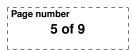
$$G_v + DF_u > 0, \quad (G_v + DF_u)^2 - 4D(F_u G_v - G_u F_v) > 0,$$

where we have used the notation

$$F_{u} = \frac{\partial F}{\partial u}\Big|_{u=u_{0}, v=v_{0}}, \quad F_{v} = \frac{\partial F}{\partial v}\Big|_{u=u_{0}, v=v_{0}},$$
$$G_{u} = \frac{\partial G}{\partial u}\Big|_{u=u_{0}, v=v_{0}}, \quad G_{v} = \frac{\partial G}{\partial v}\Big|_{u=u_{0}, v=v_{0}},$$

where (u_0, v_0) represent a homogeneous equilibrium of the system.

- (a) State the equations for a homogeneous equilibrium (u_0, v_0) of (2).
- (b) State explicitly the boundary conditions implied by "no-flux". What does this imply from a physical perspective?
- (c) Define what is meant by a pattern in this context, give the explicit mathematical form for the pattern and describe what the conditions (3) enforce.
- (d) Describe (roughly) the effect on the pattern of varying the domain size.



5. Breakout model Consider the following breakout type for a population u (a genetically modified spruce budworm),

Exam code

MATH3171-WE01

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -u^2(\beta - 2u)\gamma + u^2\left[\gamma\frac{\beta - 2u}{u} + (u - 1)(u - \beta)\right] - u\gamma^2(u - 1).$$

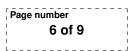
Here $\beta, \gamma > 0$ are positive constants.

- (a) What is the maximum possible number of equilibria of this system?
- (b) Show that there are always four physically valid equilibria.
- (c) If $\gamma < 1$ and $\beta > 1$, show that the stable equilibria are $u = \gamma$ and $u = \gamma + \beta$.
- 6. Chemotaxis and Slime mould Consider the following generic model for bacteria in a semi-solid medium

$$\begin{split} &\frac{\partial n}{\partial t} = D_n \nabla^2 n - \alpha \nabla \left[\frac{n}{(1+c)^2} \nabla c \right] + \rho n \left(\delta \frac{s^2}{1+s^2} - n \right), \\ &\frac{\partial c}{\partial t} = D_c \nabla^2 c + \beta s \frac{n^2}{\mu + n^2} - nc, \\ &\frac{\partial s}{\partial t} = D_s \nabla^2 s - \kappa n \frac{s^2}{1+s^2}, \end{split}$$

where $D_n, D_c, D_s, \alpha, \rho, \delta, \beta, \mu$ and κ are positive constants, n is the bacteria concentration, c is the chemotaxant concentration, and s is the nutrient concentration.

- (a) Describe all the terms in this model.
- (b) What is the required condition relating the diffusion constants D_n and D_c and why?
- (c) We assume $\kappa \ll 1$ so that the nutrition *s* remains constant, and we further assume it is homogeneously distributed. Find the permissible homogeneous equilibria n_0, c_0 for the system.



ſ	Exam code	ר - ו
1	MATH3171-WE01	
- i		- i
1		- I

SECTION B

7. **Turing Analysis** Consider the following generic chemotactic reaction-diffusion system:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla^2 u + \gamma F(u, v) + \alpha \nabla \cdot \left(\chi(u, v) \nabla v \right), \\ \frac{\partial v}{\partial t} &= D \nabla^2 v + \gamma G(u, v), \end{aligned}$$

where D > 0 is the ratio of diffusion constants of v and u and $\gamma > 0$, $\alpha > 0$ are constants. We assume nothing can leave or enter the system and that the domain V is Cartesian.

- (a) State explicitly the boundary conditions of this system.
- (b) State the equations which would determine the homogeneous equilibria (u_0, v_0) of this system.
- (c) Carry out a Turing analysis for this system to show there will be a growing inhomogeneous pattern when

$$F_u + G_v < 0, \quad F_u G_v - F_v G_u > 0,$$

$$G_v + DF_u - \alpha \chi_0 G_u > 0, \quad [G_v + DF_u - \alpha \chi_0 G_u]^2 - 4D(F_u G_v - F_v G_u) \ge 0.$$

Here we have used the notation

$$F_{u} = \frac{\partial F}{\partial u}\Big|_{u=u_{0}, v=v_{0}}, \quad F_{v} = \frac{\partial F}{\partial v}\Big|_{u=u_{0}, v=v_{0}},$$
$$G_{u} = \frac{\partial G}{\partial u}\Big|_{u=u_{0}, v=v_{0}}, \quad G_{v} = \frac{\partial G}{\partial v}\Big|_{u=u_{0}, v=v_{0}},$$
$$\chi_{0} = \chi(u_{0}, v_{0}).$$

- (d) Show that it is possible to satisfy these inequalities if G_v and F_u have the same sign. State a necessary condition on α , G_u , G_v and χ_0 required for this possibility.
- (e) Now we choose χ to be constant (χ_0) and assume $0 < \alpha \chi_0 < 1$. We set

$$F = u \left(\frac{k_1 v}{k_2 + v} - u\right), \quad G = (u - 1)(v - 1),$$

where the constants k_1 and k_2 are positive. Find the three homogeneous equilibria of the system. Now consider equilibrium for which $u_0 = 1$. Show that we require $v_0 < 1$ for the possibility that the first three conditions,

$$F_u + G_v < 0, \quad F_u G_v - F_v G_u > 0,$$

$$G_v + DF_u - \alpha \chi_0 G_u > 0,$$

can be satisfied.

MATH3171-WE01

Exam code

8. Species interaction Consider the following system for modelling competitive species u and v,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u\left(\frac{k_1v}{k_2+v} - u\right),$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = v(1-v) - \gamma uv,$$

where k_1 is a real constant, and the constants k_2 and γ are positive.

- (a) Describe the self-competitive terms and interaction terms present in the system.
- (b) Find all four equilibria of the system.
- (c) For each equilibrium state the additional conditions (if any) on k_1, k_2 and γ required for these solutions to be physically valid.
- (d) Show that the matrix A_1 of the system, linearised about an equilibrium (u_0, v_0) , is

$$A_1 = \begin{pmatrix} \frac{k_1 v_0}{k_2 + v_0} - 2u_0 & u_0 \begin{bmatrix} \frac{k_1}{k_2 + v_0} - \frac{k_1 v_0}{(k_2 + v_0)^2} \end{bmatrix} \\ -\gamma v_0 & 1 - 2v_0 - \gamma u_0 \end{pmatrix}.$$

(e) Assess the stability of those two equilibria in which at least one of the populations is extinct. 9. Spiral Waves in reaction-diffusion systems Consider the following reactiondiffusion system for the scalar densities u and v

Exam code

MATH3171-WE01

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_1 \nabla^2 u + \alpha \lambda^3 u - \xi v - u^2 v, \\ \frac{\partial v}{\partial t} &= D_2 \nabla^2 v + \xi u + \gamma \lambda^3 v - \mu u^2 v, \end{aligned}$$

where $\lambda \in \mathbb{R}$ and $\gamma, \mu, D_1, D_2, \alpha, \xi > 0$ are all constants. We assume the system is contained in a 2-D Cartesian box $[0, L] \times [0, L]$ with coordinate system (x_1, x_2) . Both u and v are assumed to satisfy flux-less boundary conditions

$$\frac{\partial u}{\partial x_i} = 0$$
 for $x_i = 0$ and L , $\frac{\partial v}{\partial x_i} = 0$ for $x_i = 0$ and L ,

for i = 1, 2.

- (a) Consider the homogeneous equilibrium $u_0 = v_0 = 0$. Show that the smallest value of λ at which this equilibrium exhibits a Hopf bifurcation is $\lambda = 0$.
- (b) Set $\lambda = \epsilon$ (we vary λ from zero by some small amount). Assume that u and v take the form $A(\tau, \beta)P(t^*)$, with $t = (1 + \omega\epsilon)t^*$, $\tau = \epsilon^2 t$, $\beta = \epsilon x$, $\omega \approx \omega_0 \epsilon + \omega_1 \epsilon^2 + \dots$ and P a periodic function. We expand u and v as

$$u(\tau,\beta,t^*) = F_1(\tau,\beta,t^*)\epsilon + F_2(\tau,\beta,t^*)\epsilon^2 + F_3(\tau,\beta,t^*)\epsilon^3 + \mathcal{O}(\epsilon^4),$$

$$v(\tau,\beta,t^*) = G_1(\tau,\beta,t^*)\epsilon + G_2(\tau,\beta,t^*)\epsilon^2 + G_3(\tau,\beta,t^*)\epsilon^3 + \mathcal{O}(\epsilon^4).$$

Show that

$$F_1 = A(\tau, \beta) \cos \left[\xi t^* + \Theta(\tau, \beta)\right], \quad G_1 = A(\tau, \beta) \sin \left[\xi t^* + \Theta(\tau, \beta)\right].$$

(c) Show that the functions A and Θ must satisfy

$$\frac{\partial A}{\partial \tau} = D \left[\nabla_{\beta}^{2} A - A (\nabla_{\beta} \Theta)^{2} \right] - \frac{\mu A^{3}}{8},$$
$$\frac{\partial \Theta}{\partial \tau} = D \left(\frac{2 \nabla_{\beta} \Theta \cdot \nabla_{\beta} A}{A} + \nabla_{\beta}^{2} \Theta \right) + \omega_{0} \xi + \frac{A^{2}}{8}$$

You may use the identities

$$-\frac{\partial F_1}{\partial \tau} + D\nabla_\beta^2 F_1 - \omega_0 \xi G_1 = \left[-\frac{\partial A}{\partial \tau} \frac{1}{A} + D\left(\frac{\nabla_\beta^2 A}{A} - (\nabla_\beta \Theta)^2\right) \right] F_1 + \left[\frac{\partial \Theta}{\partial \tau} - D\frac{2\nabla_\beta \Theta \cdot \nabla_\beta A}{A} - D\nabla_\beta^2 \Theta - \omega_0 \xi \right] G_1$$

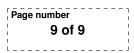
and

$$\cos^{3}(x) = \frac{1}{4} [3\cos(x) + \cos(3x)],$$

$$\cos^{2}(x)\sin(x) = \frac{1}{4} [\sin(x) + \sin(3x)],$$

$$\cos(x)\sin^{2}(x) = \frac{1}{4} [\cos(x) - \cos(3x)].$$

CONTINUED



10. **Peak mode analysis** Consider the following one-dimensional chemotactic slime mould model,

$$\begin{split} \frac{\partial n}{\partial t} &= D \frac{\partial^2 n}{\partial x^2} - \alpha \frac{\partial}{\partial x} \left[n \frac{\partial c}{\partial x} \right],\\ \frac{\partial c}{\partial t} &= \frac{\partial^2 c}{\partial x^2} + \gamma c \frac{n^2}{\mu + n^2}, \end{split}$$

where α, μ are positive constants and $\gamma \in \mathbb{R}$.

(a) Let $n = n_0$ be a spatially homogeneous equilibrium state for n. Suppose n is in this equilibrium state. If we further assume that c has a spatially homogeneous profile, which initially takes a value c_0 , then show that

$$c = c_0 e^{rt}$$
, where $r = \gamma \frac{n_0^2}{\mu + n_0^2}$

(b) Consider solutions of the form

$$n(x,t) = n_0 + \epsilon f(t) \sum_k e^{ikx}, \quad c(x,t) = c_0 e^{rt} + \epsilon g(t) \sum_k e^{ikx},$$

where $\epsilon \ll 1$. Show that, for each k, the $\mathcal{O}(\epsilon)$ equations are:

$$\frac{\partial f}{\partial t} = -Dk^2 f + \alpha n_0 k^2 g,$$

$$\frac{\partial g}{\partial t} = -\left(k^2 - r\right)g + \frac{2n_0 c_0}{\mu + n_0^2}\left(\gamma - r\right)e^{rt}f.$$
(4)

Exam code

MATH3171-WE01

(c) Show that the system (4) can be reduced to a single ordinary differential equation for f in the form

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + a(k)\frac{\mathrm{d}f}{\mathrm{d}t} + \left[b(k)\mathrm{e}^{rt} + c(k)\right]f = 0,$$

where

$$a(k) = [Dk^{2} + (k^{2} - r)], \quad b(k) = \alpha n_{0}\beta (r - \gamma) k^{2}, \quad \beta = \frac{2n_{0}c_{0}}{\mu + n_{0}^{2}},$$
$$c(k) = (k^{2} - r) Dk^{2}.$$

(d) One technique for assessing the behaviour of each mode could be to make the assumption that the coefficients of the equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + a(k)\frac{\mathrm{d}f}{\mathrm{d}t} + \left[b(k)\mathrm{e}^{rt} + c(k)\right]f = 0$$

vary at a rate slower than the function f and its derivatives. Why is this assumption incorrect in this case?

(e) We now assume $r/\gamma = 1$ and consider the problem on a domain of length L with no-flux boundary conditions. State a condition for which all inhomogeneous growth modes will be stable.