Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH3171-WE01

Title:
Mathematical Biology III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. **Fourier transform and Green's method** Consider the following P.D.E. and its fundamental problem:

$$\frac{\partial^2 c_f}{\partial t^2} = D\frac{\partial^2 c_f}{\partial x^2}, \quad c_f(x,0) = \delta(x), \quad \frac{\partial c_f}{\partial t}(x,0) = 0, \quad c_f(\pm\infty, t) = 0. \tag{1}$$

Here $\delta(x)$ is the Dirac delta distribution and $D > 0$. We define the Fourier Transform as

$$F[c](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} c(x)\mathrm{d}x,$$

and its inverse as

$$F^{-1}[\hat{c}](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \hat{c}(k)\mathrm{d}k.$$

(a) Show that (1) implies

$$F[c_f](k) = \frac{1}{\sqrt{2\pi}}\cos(k\sqrt{D}t).$$

(b) Using Green's method solve the problem

$$\frac{\partial^2 c}{\partial t^2} = D\frac{\partial^2 c}{\partial x^2}, \quad c(x,0) = g(x), \quad \frac{\partial c}{\partial t}(x,0) = 0, \quad c(\pm\infty, t) = 0,$$
$$g(x) = \begin{cases} 0 & x < -a, \\ 1 & -a \le x \le a, \\ 0 & x > a \end{cases}$$

and describe briefly the changing shape of the distribution over time. You may use the result

$$F[\cos(ax)](k) = \sqrt{\frac{\pi}{2}}\left(\delta(k-a) + \delta(k+a)\right).$$

2. **P.D.E. stability** Consider the following P.D.E. and boundary conditions

$$\frac{\partial^2 y}{\partial x^2} + (1-y)\frac{\partial y}{\partial x} + \alpha y^3 = \frac{\partial y}{\partial t}, \quad y(0,t) = 0, \quad y(L,t) = 0,$$

where $\alpha > 0$ is a constant. There is a homogeneous equilibrium $y = 0$.

(a) Assess the stability of this equilibrium.

(b) How would this conclusion have changed if the problem had instead been

$$\frac{\partial^2 y}{\partial x^2} + (1-y)\frac{\partial y}{\partial x} + \alpha y = \frac{\partial y}{\partial t}, \quad y(0,t) = 0, \quad y(L,t) = 0\,?$$

State any conditions on domain length L, dependent on the value of the parameter α, which will ensure equilibrium $y = 0$ is stable.

3. **Limit Cycles and the Hopf-bifurcation** Consider the following activator-inhibitor system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a - bu + \frac{u^2}{v},$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = u^2 - v,$$

where $a, b > 0$ are **real** constants.

(a) Find the single equilibrium of the system.

(b) Show that this equilibrium exhibits a Hopf-bifurcation when

$$b = \frac{a+1}{1-a},$$

and $a \neq 1$.

(c) Find a quadratic equation for b which gives the boundary between oscillatory and non-oscillatory behaviour of the linearised system. To simplify your algebra make the substitution $\gamma = \frac{2}{a+1} - 1$.

4. **Turing analysis** Consider the following **non-dimensionalised** reaction-diffusion system for scalar densities u and v

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla^2 u + \gamma F(u,v), \\ \frac{\partial v}{\partial t} &= D\nabla^2 v + \gamma G(u,v),\end{aligned} \tag{2}$$

where $D > 0$ and $\gamma > 0$ are constants. We assume **no-flux** boundary conditions on a Cartesian domain $[0, L_1] \times [0, L_2]$ with coordinates (x_1, x_2). The Turing conditions for pattern formation are

$$\begin{aligned}&F_u + G_v < 0, \quad F_u G_v - G_u F_v > 0, \\ &G_v + DF_u > 0, \quad (G_v + DF_u)^2 - 4D(F_u G_v - G_u F_v) > 0,\end{aligned} \tag{3}$$

where we have used the notation

$$\begin{aligned}F_u &= \left.\frac{\partial F}{\partial u}\right|_{u=u_0, v=v_0}, \quad F_v = \left.\frac{\partial F}{\partial v}\right|_{u=u_0, v=v_0}, \\ G_u &= \left.\frac{\partial G}{\partial u}\right|_{u=u_0, v=v_0}, \quad G_v = \left.\frac{\partial G}{\partial v}\right|_{u=u_0, v=v_0},\end{aligned}$$

where (u_0, v_0) represent a homogeneous equilibrium of the system.

(a) State the equations for a homogeneous equilibrium (u_0, v_0) of (2).

(b) State explicitly the boundary conditions implied by "no-flux". What does this imply from a physical perspective?

(c) Define what is meant by a pattern in this context, give the explicit mathematical form for the pattern and describe what the conditions (3) enforce.

(d) Describe (roughly) the effect on the pattern of varying the domain size.

5. **Breakout model** Consider the following breakout type for a population u (a genetically modified spruce budworm),

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -u^2(\beta - 2u)\gamma + u^2\left[\gamma\frac{\beta - 2u}{u} + (u-1)(u-\beta)\right] - u\gamma^2(u-1).$$

Here $\beta, \gamma > 0$ are positive constants.

(a) What is the maximum possible number of equilibria of this system?

(b) Show that there are always four physically valid equilibria.

(c) If $\gamma < 1$ and $\beta > 1$, show that the stable equilibria are $u = \gamma$ and $u = \gamma + \beta$.

6. **Chemotaxis and Slime mould** Consider the following generic model for bacteria in a semi-solid medium

$$\frac{\partial n}{\partial t} = D_n\nabla^2 n - \alpha\nabla\left[\frac{n}{(1+c)^2}\nabla c\right] + \rho n\left(\delta\frac{s^2}{1+s^2} - n\right),$$
$$\frac{\partial c}{\partial t} = D_c\nabla^2 c + \beta s\frac{n^2}{\mu + n^2} - nc,$$
$$\frac{\partial s}{\partial t} = D_s\nabla^2 s - \kappa n\frac{s^2}{1+s^2},$$

where $D_n, D_c, D_s, \alpha, \rho, \delta, \beta, \mu$ and κ are positive constants, n is the bacteria concentration, c is the chemotaxant concentration, and s is the nutrient concentration.

(a) Describe all the terms in this model.

(b) What is the required condition relating the diffusion constants D_n and D_c and why?

(c) We assume $\kappa \ll 1$ so that the nutrition s remains constant, and we further assume it is homogeneously distributed. Find the permissible homogeneous equilibria n_0, c_0 for the system.

SECTION B

7. **Turing Analysis** Consider the following generic chemotactic reaction-diffusion system:

$$\frac{\partial u}{\partial t} = \nabla^2 u + \gamma F(u,v) + \alpha \nabla \cdot (\chi(u,v)\nabla v),$$
$$\frac{\partial v}{\partial t} = D\nabla^2 v + \gamma G(u,v),$$

where $D > 0$ is the ratio of diffusion constants of v and u and $\gamma > 0$, $\alpha > 0$ are constants. We assume nothing can leave or enter the system and that the domain V is Cartesian.

(a) State explicitly the boundary conditions of this system.

(b) State the equations which would determine the homogeneous equilibria (u_0, v_0) of this system.

(c) Carry out a Turing analysis for this system to show there will be a growing inhomogeneous pattern when

$$F_u + G_v < 0, \quad F_u G_v - F_v G_u > 0,$$
$$G_v + DF_u - \alpha\chi_0 G_u > 0, \quad [G_v + DF_u - \alpha\chi_0 G_u]^2 - 4D(F_u G_v - F_v G_u) \geq 0.$$

Here we have used the notation

$$F_u = \left.\frac{\partial F}{\partial u}\right|_{u=u_0,v=v_0}, \quad F_v = \left.\frac{\partial F}{\partial v}\right|_{u=u_0,v=v_0},$$
$$G_u = \left.\frac{\partial G}{\partial u}\right|_{u=u_0,v=v_0}, \quad G_v = \left.\frac{\partial G}{\partial v}\right|_{u=u_0,v=v_0},$$
$$\chi_0 = \chi(u_0, v_0).$$

(d) Show that it is possible to satisfy these inequalities if G_v and F_u have the same sign. State a necessary condition on α, G_u, G_v and χ_0 required for this possibility.

(e) Now we choose χ to be constant (χ_0) and assume $0 < \alpha\chi_0 < 1$. We set

$$F = u\left(\frac{k_1 v}{k_2 + v} - u\right), \quad G = (u-1)(v-1),$$

where the constants k_1 and k_2 are positive. Find the three homogeneous equilibria of the system. Now consider equilibrium for which $u_0 = 1$. Show that we require $v_0 < 1$ for the possibility that the first three conditions,

$$F_u + G_v < 0, \quad F_u G_v - F_v G_u > 0,$$
$$G_v + DF_u - \alpha\chi_0 G_u > 0,$$

can be satisfied.

8. **Species interaction** Consider the following system for modelling competitive species u and v,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u\left(\frac{k_1 v}{k_2 + v} - u\right),$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = v(1 - v) - \gamma uv,$$

where k_1 is a real constant, and the constants k_2 and γ are positive.

(a) Describe the self-competitive terms and interaction terms present in the system.

(b) Find all four equilibria of the system.

(c) For each equilibrium state the additional conditions (if any) on k_1, k_2 and γ required for these solutions to be physically valid.

(d) Show that the matrix A_1 of the system, linearised about an equilibrium (u_0, v_0), is

$$A_1 = \begin{pmatrix} \frac{k_1 v_0}{k_2+v_0} - 2u_0 & u_0\left[\frac{k_1}{k_2+v_0} - \frac{k_1 v_0}{(k_2+v_0)^2}\right] \\ -\gamma v_0 & 1 - 2v_0 - \gamma u_0 \end{pmatrix}.$$

(e) Assess the stability of those two equilibria in which at least one of the populations is extinct.

9. **Spiral Waves in reaction-diffusion systems** Consider the following reaction-diffusion system for the scalar densities u and v

$$\frac{\partial u}{\partial t} = D_1 \nabla^2 u + \alpha\lambda^3 u - \xi v - u^2 v,$$
$$\frac{\partial v}{\partial t} = D_2 \nabla^2 v + \xi u + \gamma\lambda^3 v - \mu u^2 v,$$

where $\lambda \in \mathbb{R}$ and $\gamma, \mu, D_1, D_2, \alpha, \xi > 0$ are all constants. We assume the system is contained in a 2-D Cartesian box $[0, L] \times [0, L]$ with coordinate system (x_1, x_2). Both u and v are assumed to satisfy flux-less boundary conditions

$$\frac{\partial u}{\partial x_i} = 0 \text{ for } x_i = 0 \text{ and } L, \quad \frac{\partial v}{\partial x_i} = 0 \text{ for } x_i = 0 \text{ and } L,$$

for $i = 1, 2$.

(a) Consider the homogeneous equilibrium $u_0 = v_0 = 0$. Show that the smallest value of λ at which this equilibrium exhibits a Hopf bifurcation is $\lambda = 0$.

(b) Set $\lambda = \epsilon$ (we vary λ from zero by some small amount). Assume that u and v take the form $A(\tau, \beta)P(t^*)$, with $t = (1 + \omega\epsilon)t^*$, $\tau = \epsilon^2 t$, $\beta = \epsilon x$, $\omega \approx \omega_0\epsilon + \omega_1\epsilon^2 + ...$ and P a periodic function. We expand u and v as

$$u(\tau, \beta, t^*) = F_1(\tau, \beta, t^*)\epsilon + F_2(\tau, \beta, t^*)\epsilon^2 + F_3(\tau, \beta, t^*)\epsilon^3 + \mathcal{O}(\epsilon^4),$$
$$v(\tau, \beta, t^*) = G_1(\tau, \beta, t^*)\epsilon + G_2(\tau, \beta, t^*)\epsilon^2 + G_3(\tau, \beta, t^*)\epsilon^3 + \mathcal{O}(\epsilon^4).$$

Show that

$$F_1 = A(\tau, \beta)\cos\left[\xi t^* + \Theta(\tau, \beta)\right], \quad G_1 = A(\tau, \beta)\sin\left[\xi t^* + \Theta(\tau, \beta)\right].$$

(c) Show that the functions A and Θ must satisfy

$$\frac{\partial A}{\partial \tau} = D\left[\nabla^2_\beta A - A(\nabla_\beta \Theta)^2\right] - \frac{\mu A^3}{8},$$
$$\frac{\partial \Theta}{\partial \tau} = D\left(\frac{2\nabla_\beta\Theta \cdot \nabla_\beta A}{A} + \nabla^2_\beta\Theta\right) + \omega_0\xi + \frac{A^2}{8}.$$

You may use the identities

$$-\frac{\partial F_1}{\partial \tau} + D\nabla^2_\beta F_1 - \omega_0\xi G_1 =$$
$$\left[-\frac{\partial A}{\partial \tau}\frac{1}{A} + D\left(\frac{\nabla^2_\beta A}{A} - (\nabla_\beta\Theta)^2\right)\right] F_1 \;\; + \left[\frac{\partial \Theta}{\partial \tau} - D\frac{2\nabla_\beta\Theta \cdot \nabla_\beta A}{A} - D\nabla^2_\beta\Theta - \omega_0\xi\right] G_1,$$

and

$$\cos^3(x) = \frac{1}{4}\left[3\cos(x) + \cos(3x)\right],$$
$$\cos^2(x)\sin(x) = \frac{1}{4}\left[\sin(x) + \sin(3x)\right],$$
$$\cos(x)\sin^2(x) = \frac{1}{4}\left[\cos(x) - \cos(3x)\right].$$

10. **Peak mode analysis** Consider the following one-dimensional chemotactic slime mould model,

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} - \alpha\frac{\partial}{\partial x}\left[n\frac{\partial c}{\partial x}\right],$$
$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + \gamma c\frac{n^2}{\mu + n^2},$$

where α, μ are positive constants and $\gamma \in \mathbb{R}$.

(a) Let $n = n_0$ be a spatially homogeneous equilibrium state for n. Suppose n is in this equilibrium state. If we further assume that c has a spatially homogeneous profile, which initially takes a value c_0, then show that

$$c = c_0 \mathrm{e}^{rt}, \text{ where } r = \gamma\frac{n_0^2}{\mu + n_0^2}.$$

(b) Consider solutions of the form

$$n(x,t) = n_0 + \epsilon f(t)\sum_k \mathrm{e}^{ikx}, \quad c(x,t) = c_0\mathrm{e}^{rt} + \epsilon g(t)\sum_k \mathrm{e}^{ikx},$$

where $\epsilon \ll 1$. Show that, for each k, the $\mathcal{O}(\epsilon)$ equations are:

$$\frac{\partial f}{\partial t} = -Dk^2 f + \alpha n_0 k^2 g, \tag{4}$$
$$\frac{\partial g}{\partial t} = -\left(k^2 - r\right) g + \frac{2n_0 c_0}{\mu + n_0^2}\left(\gamma - r\right)\mathrm{e}^{rt} f.$$

(c) Show that the system (4) can be reduced to a single ordinary differential equation for f in the form

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + a(k)\frac{\mathrm{d}f}{\mathrm{d}t} + \left[b(k)\mathrm{e}^{rt} + c(k)\right] f = 0,$$

where

$$a(k) = \left[Dk^2 + \left(k^2 - r\right)\right], \quad b(k) = \alpha n_0 \beta\left(r - \gamma\right)k^2, \quad \beta = \frac{2n_0 c_0}{\mu + n_0^2},$$
$$c(k) = \left(k^2 - r\right)Dk^2.$$

(d) One technique for assessing the behaviour of each mode could be to make the assumption that the coefficients of the equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + a(k)\frac{\mathrm{d}f}{\mathrm{d}t} + \left[b(k)\mathrm{e}^{rt} + c(k)\right] f = 0,$$

vary at a rate slower than the function f and its derivatives. Why is this assumption incorrect in this case?

(e) We now assume $r/\gamma = 1$ and consider the problem on a domain of length L with no-flux boundary conditions. State a condition for which all inhomogeneous growth modes will be stable.

