Durham University

EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May | 2017 | MATH3281-WE01 |

**Title:** Topology III

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| Time Allowed: | 3 hours | |
| Additional Material provided: | None | |
| Materials Permitted: | None | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |
| Visiting Students may use dictionaries: Yes | | |

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| Instructions to Candidates: | Credit will be given for:<br>the best **FOUR** answers from Section A<br>and the best **THREE** answers from Section B.<br>Questions in Section B carry **TWICE** as many marks as those in Section A. |

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| **Revision:** | |

ED01/2017

**The following conventions hold in this paper**

- $\mathbb{Z}$ denotes the additive group of all integer numbers with the discrete topology.
- $\mathbb{R}$ denotes the space of all real numbers with the Euclidean topology.
- $\mathbb{C}$ denotes the space of all complex numbers with the Euclidean topology.
- $\mathbb{R}^n$ denotes the real $n$-dimensional space with the Euclidean topology.

## SECTION A

1. (i) State the definition of a topological space.

   (ii) Let $X = \{a, b, c\}$. List six topologies on $X$ so that no two of them are homeomorphic. You do not need to justify why they are not homeomorphic.

2. Let $X$ and $Y$ be topological spaces.

   (i) State the definition of the product topology on $X \times Y$.

   (ii) Let $Z$ be a topological space. Show that a function $f\colon Z \to X \times Y$ is continuous if and only if $p_X \circ f\colon Z \to X$ and $p_Y \circ f\colon Z \to Y$ are continuous, where $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ are the projections.

3. (a) State the definition of compactness for a topological space.

   (b) Using this definition, show that $\mathbb{R}$ with the standard topology is not compact.

   (c) Give a topology on $\mathbb{R}$ which is compact.

4. (i) State the classification theorem for closed surfaces without boundary. State the relationship between adding handles and adding cross caps to a surface.

   (ii) If $S$ is a closed surface and $C$ is a simple closed curve in $S$, write $\Theta(C)$ for the *thickening* of $C$ in $S$. In the context of $S$ triangulated by some finite simplicial complex $K$, and $C$ being a curve contained in the 1-skeleton of $K$, give a definition of $\Theta(C)$. Explain in general why if $S$ is non-orientable then there must be some $C$ such that $\Theta(C)$ is homeomorphic to a Möbius band.

5. (a) In this question, denote by $S_r$ the circle of complex numbers of modulus $r > 0$, and let $f\colon \mathbb{C} \to \mathbb{C}$ be a continuous function on the complex plane. Denote by $L_r$ the loop in the complex plane given by the image of $f(S_r)$. If the origin $0 \in \mathbb{C}$ is not in the image of $L_r$, explain what is meant by $w(L_r, 0)$, the *winding number* of $L_r$ about 0.

   (b) Now let $f$ be given by the function $z \mapsto z^2 - 2z + 1$. Letting $r$ range over all positive real numbers, what are the different values $w(L_r, 0)$ can take (when it is defined)? You may use any facts about winding numbers from lectures provided you state what you are using.

6. Let $X$ be the space defined as a quotient of the square $[0,1]\times[0,1]$ by the equivalence relation given by

$$(0,t) \;\sim\; (t,1) \;\sim\; (1,1-t) \;\sim\; (1-t,0)$$

Give a triangulation of $X$ and use it to compute the fundamental group $\pi_1(X)$. You may use any technique from lectures, but you should briefly explain your calculations.

## SECTION B

7. Let $X$ be a non-empty topological space. Define a relation $\sim$ on $X$ by $x \sim y$ if there exists a path in $X$ from $x$ to $y$. For $x \in X$ let

$$P_x = \{y \in X \mid y \sim x\}.$$

(a) Show that $\sim$ is an equivalence relation.

(b) Show that $X$ is path connected if and only if the quotient space $X/\sim$ is a point.

(c) Show that $P_x$ is connected.

(d) Assume that $X$ is an open subset of $\mathbb{R}^n$. Show that $P_x$ is an open subset of $X$ for every $x \in X$.

(e) Let $X \subset \mathbb{R}^n$ be a connected, open subset. Show that $X$ is path connected.

8. For $n \in \mathbb{Z}$, $a \in \mathbb{R}$ and $z \in S^1 = \{w \in \mathbb{C} \mid |w|^2 = 1\}$ define

$$n \bullet z = e^{2\pi i a n} \cdot z$$

where the right side is multiplication of complex numbers.

(a) Show that $\bullet \colon \mathbb{Z} \times S^1 \to S^1$ defines an action of $\mathbb{Z}$ on $S^1$ for every $a \in \mathbb{R}$.

(b) Determine the values of $a \in \mathbb{R}$ for which this action is the trivial action.

(c) Show that for rational $a \in \mathbb{Q}$ the quotient space $S^1/\mathbb{Z}$ is homeomorphic to $S^1$.

(d) Show that for irrational $a \in \mathbb{R} - \mathbb{Q}$ each orbit $\mathbb{Z}z$ has a limit point in $S^1$.

(e) Show that for irrational $a \in \mathbb{R} - \mathbb{Q}$ each orbit $\mathbb{Z}z$ has every point $w \in S^1$ as a limit point.

9. (i) Define what it means to say that two maps $f, g \colon X \to Y$ are *homotopic*, and what it means to say that two spaces $A$ and $B$ are *homotopy equivalent*. Prove that homotopy equivalence of spaces is an equivalence relation on spaces (you may assume that homotopy of maps is an equivalence relation on maps).

(ii) Prove that the open 2-disc $E^2 = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$ is homotopy equivalent to the closed 2-disc $D^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leqslant 1\}$.

(iii) (a) Suppose the finite, path connected simplicial complex $X$ is the union of two subcomplexes $A$ and $B$, and denote the intersection $A \cap B$ by $C$. State a relationship between the Euler characteristics of $X$, $A$, $B$ and $C$. State van Kampen's theorem relating the fundamental groups of these spaces, remembering to state any further hypotheses about these spaces you need for the theorem to be true.

(b) Suppose further that the subspace $C$ in (a) is homeomorphic to a circle, and bounds discs in each of $A$ and $B$. What can you now say about $\chi(X)$ and $\pi_1(X)$ in terms of $\chi$ and $\pi_1$ of the subspaces? If you use van Kampen's theorem, make sure you check any hypotheses needed.

10. (i) State and prove Euler's theorem about graphs embedded on a sphere. You may assume any results about the Euler characteristic of a tree from lectures.

(ii) Suppose $K$ is a finite simplicial complex which triangulates a closed surface $S$. What conclusions about the triangulation can you draw from the fact that $S$ is a surface?

(iii) The sphere $S^2$ is triangulated by a simplicial complex with $V$ vertices, $E$ 1-simplices and $F$ 2-simplices. The triangulation has the property that at each vertex exactly $n$ edges meet. Use the Euler characteristic of $S^2$ to obtain a relation between the numbers $V$ and $n$. Hence prove that only certain values of $n$ are possible; find them all, identifying the simplicial complexes obtained.

(iv) Is it possible to have a triangulation of the torus in which at each vertex exactly 5 edges meet?