EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH3291-WE01

Title:
Partial Differential Equations III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. Consider the conservation law

$$\begin{aligned} u_t + uu_x &= 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) &= e^{-x^4} \quad \text{for } x \in \mathbb{R}. \end{aligned} \tag{1}$$

   (a) Find the largest value of $T \in \mathbb{R}$ for which this conservation law has a classical solution $u : \mathbb{R} \times [0,T) \to \mathbb{R}$.

   (b) Give a sketch of the characteristics of (1) up until time T.

   (c) State what it means for u to be a weak solution of (1).

2. Let $c, u_0 \in C^1(\mathbb{R})$, u_0 be bounded, and $u_0'(x) \to 0$ as $x \to \pm\infty$. Define

$$I := \{s \in \mathbb{R} : c(u_0(s))_s < 0\}.$$

   Assume that I is nonempty. Let $s_\mathrm{c} \in I$ satisfy

$$\frac{-1}{c'(u_0(s_\mathrm{c}))u_0'(s_\mathrm{c})} = \min_{s \in I} \frac{-1}{c'(u_0(s))u_0'(s)} =: t_\mathrm{c}.$$

   Let $\tilde{s} : \mathbb{R} \times [0, t_\mathrm{c}) \to \mathbb{R}$ be the unique function satisfying $x = \tilde{s}(x,t) + c(u_0(\tilde{s}(x,t)))t$.

   (a) State the solution of the conservation law

$$\begin{aligned} u_t + c(u)u_x &= 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,t_\mathrm{c}), \\ u(x,0) &= u_0(x) \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

   (You do not need to derive the solution.)

   (b) Define $x_\mathrm{c}(t) = s_\mathrm{c} + c(u_0(s_\mathrm{c}))t$. Prove that

$$\lim_{t \to t_\mathrm{c}^-} |u_t(x_\mathrm{c}(t), t)| = \infty.$$

   (c) Show that u satisfies the implicit equation

$$u(x,t) = u_0(x - c(u(x,t))t).$$

3. Consider Poisson's equation in one dimension with Neumann boundary conditions:

$$-u''(x) = f(x), \quad x \in (a,b), \tag{2}$$
$$u'(a) = 0, \; u'(b) = 0, \tag{3}$$

where $f \in C([a,b])$.

(a) Show that a necessary condition for (2), (3) to have a solution is

$$\int_a^b f(x)\,dx = 0. \tag{4}$$

(b) Assume that f satisfies (4). Find the unique solution of (2), (3) satisfying $u(a) = 0$. Write your solution in the form

$$u(x) = \int_a^b G(x,y) f(y)\,dy.$$

4. Consider the elliptic boundary value problem

$$-(|u'|^{p-2}u')' + \gamma u = 0, \quad x \in (0, 2\pi), \tag{5}$$
$$u(0) = u(2\pi) = 0, \tag{6}$$

where $p \geq 2$ and $\gamma \in \mathbb{R}$ are constants.

(a) Let $\gamma \geq 0$. Show that (5), (6) has a unique solution.

(b) Now consider the case $p = 2$, $\gamma < 0$. Find a sequence $\{\gamma_n\}_{n=1}^{\infty}$, with $\gamma_n < 0$ and $\gamma_n \neq \gamma_m$ for $n \neq m$, such that, for each $n \in \mathbb{N}$, the following boundary value problem has infinitely many solutions:

$$-u'' + \gamma_n u = 0, \quad x \in (0, 2\pi),$$
$$u(0) = u(2\pi) = 0.$$

5. Let $\Omega \subset \mathbb{R}^2$ be open and bounded with smooth boundary. Define $E : C^1(\overline{\Omega}) \to \mathbb{R}$ by

$$E[v] = \int_\Omega \sqrt{1 + |\nabla v|^2}\, d\boldsymbol{x}.$$

Let $g : \partial\Omega \to \mathbb{R}$ be a given smooth function and let

$$V = \{\varphi \in C^1(\overline{\Omega}) \, : \, \varphi = g \text{ on } \partial\Omega\}.$$

Suppose that $u \in C^2(\overline{\Omega}) \cap V$ minimises E over V:

$$E[u] = \min_{v \in V} E[v].$$

Show that u satisfies the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0 \quad \text{in } \Omega.$$

6. Let $c \in \mathbb{R}$, $k > 0$ be constants.

   (a) Recall that the inverse Fourier transform of a function $h \in L^1(\mathbb{R})$ is defined by

$$\check{h}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(\xi) e^{ix\xi} \, d\xi.$$

   For $t > 0$, define $f \in L^1(\mathbb{R})$ by

$$f(\xi) = \exp\left(-(k\xi^2 + ic\xi)t\right).$$

   Prove that

$$\check{f}(x) = \frac{1}{\sqrt{2kt}} \exp\left(-\frac{1}{4kt}(x - ct)^2\right).$$

   You may use the following, which you do not need to prove: For $a > 0$,

$$\int_\Gamma e^{-az^2} \, dz = \int_{-\infty}^{\infty} e^{-ay^2} \, dy = \sqrt{\frac{\pi}{a}}$$

   for all complex curves Γ of the form $\Gamma = \{x + ib : x \in (-\infty, \infty)\}$, $b \in \mathbb{R}$.

   (b) Consider the transport-diffusion equation

$$u_t + cu_x = ku_{xx} \quad \text{for } (x, t) \in \mathbb{R} \times (0, \infty)$$

   with initial condition $u(x, 0) = g(x)$ for $x \in \mathbb{R}$, where $g \in L^1(\mathbb{R}) \cap C(\mathbb{R})$. Use the Fourier transform and part (a) to derive the following solution:

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x - ct - y)^2}{4kt}\right) g(y) \, dy.$$

   You may use the following properties of the Fourier transform, which you do not need to prove:

$$\widehat{f_1 * f_2} = \sqrt{2\pi} \hat{f}_1 \hat{f}_2, \qquad \widehat{f^{(\alpha)}}(\xi) = (i\xi)^\alpha \hat{f}(\xi).$$

SECTION B

7. (i) Consider the Cauchy problem

$$xu_x(x,y) + yu_y(x,y) = 2u(x,y) \quad \text{for } x > 0,\ y \in \mathbb{R}, \tag{7}$$
$$u(x,y) = f(x,y) \quad \text{for } x^2 + y^2 = 1,\ x > 0. \tag{8}$$

(a) Use the method of characteristics to derive the solution u in terms of f.

(b) Now consider the Cauchy problem

$$xu_x(x,y) + yu_y(x,y) = 2u(x,y) \quad \text{for } x > 0,\ y \in \mathbb{R}, \tag{9}$$
$$u(x,0) = x^2 \quad \text{for } x > 0. \tag{10}$$

Use part (a) to show that the Cauchy problem (9), (10) has infinitely many solutions. Explain why this does not contradict the local existence and uniqueness theorem for first-order quasilinear PDEs.

(ii) (a) Let $u : \mathbb{R} \times [0,\infty) \to [0,\infty)$ be a smooth function satisfying the scalar conservation law

$$u_t + f(u)_x = 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty),$$

where $f(u) = \frac{1}{2}u^2$. Define $v = u^2$. Show that v satisfies the scalar conservation law

$$v_t + g(v)_x = 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty),$$

where $g(v) = \frac{2}{3}v^{\frac{3}{2}}$.

(b) Let

$$u_0(x) = \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0, \end{cases}$$

where u_l and u_r are constants with $u_l > u_r$. The weak solution of the conservation law

$$u_t + f(u)_x = 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty),$$
$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}$$

is

$$u(x,t) = \begin{cases} u_l & \text{if } x < \frac{1}{2}(u_l + u_r)t, \\ u_r & \text{if } x > \frac{1}{2}(u_l + u_r)t. \end{cases}$$

Let $v_0 = u_0^2$. Find the weak solution of the conservation law

$$v_t + g(v)_x = 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty),$$
$$v(x,0) = v_0(x) \quad \text{for } x \in \mathbb{R}.$$

Does $v = u^2$?

8. Consider the scalar conservation law

$$\begin{aligned} u_t + f(u)_x &= 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= u_0(x) \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{11}$$

where $f(u) = \frac{1}{4}u^4$ and

$$u_0(x) = \begin{cases} 0 & \text{if } x < 0, \\ 2 & \text{if } x > 0. \end{cases}$$

(a) Write down the equations of the characteristics (you do not need to derive them from first principles). Sketch the characteristics.

(b) Verify that the following is a weak solution of (11):

$$u(x,t) = \begin{cases} 0 & \text{if } x < \frac{1}{4}u_m^3 t, \\ u_m & \text{if } \frac{1}{4}u_m^3 t < x < u_m^3 t, \\ \left(\dfrac{x}{t}\right)^{\frac{1}{3}} & \text{if } u_m^3 t \le x \le 8t, \\ 2 & \text{if } x > 8t, \end{cases}$$

where u_m is any constant satisfying $0 < u_m < 2$.

(c) Does the solution in part (b) satisfy the Lax entropy condition? Justify your answer.

(d) Find a class of weak solutions of (11) with two rarefaction waves and one shock.

9. Let $\Omega \subset \mathbb{R}^2$ be open and bounded. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$-\Delta u + \boldsymbol{b} \cdot \nabla u = f \quad \text{in } \Omega,$$

where $\boldsymbol{b} : \Omega \to \mathbb{R}^2$, $f : \Omega \to \mathbb{R}$ are continuous.

(a) Assume that $f < 0$. Prove the following weak maximum principle:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Hint: You cannot prove this using a mean-value formula. Imitate the proof of the weak maximum principle for the heat equation.

(b) Now assume $\boldsymbol{b} = 0$ and $f \leq 0$. Use part (a) to prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Hint: Consider functions of the form $u_\varepsilon(\boldsymbol{x}) = u(\boldsymbol{x}) - \varepsilon(R^2 - |\boldsymbol{x}|^2)$ for $\varepsilon > 0$ and $R > 0$ such that $\Omega \subset B_R(\mathbf{0})$.

(c) Now assume $\boldsymbol{b} = 0$ and $f > 0$. Give an example of $\Omega \subset \mathbb{R}^2$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and $f \in C(\Omega)$ such that

$$\max_{\overline{\Omega}} u \neq \max_{\partial\Omega} u.$$

(d) For $i \in \{1, 2\}$, let $u_i \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{aligned} -\Delta u_i + \boldsymbol{b} \cdot \nabla u_i &= f_i \quad \text{in } \Omega, \\ u_i &= g_i \quad \text{on } \partial\Omega, \end{aligned}$$

where $\boldsymbol{b} : \Omega \to \mathbb{R}^2$, $f_i : \Omega \to \mathbb{R}$, $g_i : \partial\Omega \to \mathbb{R}$ are continuous.

Show that if $f_2 > f_1$ and $g_2 > g_1$, then $u_2 > u_1$.

10. Let $k > 0$ be a constant and let $u : [a,b] \times [0,\infty) \to \mathbb{R}$ be a smooth function satisfying the heat equation

$$\begin{aligned} u_t(x,t) - k u_{xx}(x,t) &= f(x) \quad \text{for } (x,t) \in (a,b) \times (0,\infty), \\ u(x,0) &= u_0(x) \quad \text{for } x \in (a,b), \\ u_x(a,t) = u_x(b,t) &= 0 \quad \text{for } t \in [0,\infty), \end{aligned}$$

where u_0 and f are smooth functions and $\int_a^b f(x)\,dx = 0$. Let $v : [a,b] \to \mathbb{R}$ be the unique solution of

$$\begin{gathered} -k v_{xx}(x) = f(x) \quad \text{for } x \in (a,b), \\ v_x(a) = v_x(b) = 0, \\ \int_a^b v(x)\,dx = \int_a^b u_0(x)\,dx. \end{gathered}$$

Define $w(x,t) = u(x,t) - v(x)$.

(a) Prove the following version of the Grönwall inequality: If $E : [0,\infty) \to \mathbb{R}$ satisfies $\dot{E} \leq -\lambda E$ for some constant $\lambda \in \mathbb{R}$, then $E(t) \leq e^{-\lambda t} E(0)$.

(b) Prove that w satisfies

$$\frac{d}{dt} \int_a^b w^2(x,t)\,dx = -2k \int_a^b w_x^2(x,t)\,dx.$$

(c) Prove that

$$\int_a^b u(x,t)\,dx = \int_a^b u_0(x)\,dx \quad \forall\, t \geq 0.$$

(d) Prove that $w \to 0$ in $L^2([a,b])$ as $t \to \infty$.

(e) Similarly, it can be shown that $w_x \to 0$ in $L^2([a,b])$ as $t \to \infty$ (you do not need to show this). Prove that $w \to 0$ in $L^\infty([a,b])$ as $t \to \infty$.