Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH3391-WE01

Title:
Quantum Information III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. A measurement operator $\hat{A}$ has eigenvalues $n \in \{0, 1, 2, 3\}$ with normalised eigenstates $|n\rangle$.

   (a) If the system is in the pure state

   $$N\left(\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{i}{\sqrt{2}}|2\rangle - \frac{i}{\sqrt{2}}|3\rangle\right)$$

   for some $N \in \mathbb{C}$, what are the possible outcomes and associated probabilities when measuring A.

   (b) Calculate the expectation value of A in the above state.

   (c) Describe what a mixed state is in general, and describe the specific mixed state which has the same possible outcomes and associated probabilities as for the pure state $|\psi\rangle = \tilde{N}(|0\rangle + |1\rangle)$, for $\tilde{N} \in \mathbb{C}$, when measuring A.

   Describe one way to distinguish a collection of a large number of such mixed states from a large number of pure states $|\psi\rangle$.

2. Consider a single qubit Hilbert space with orthonormal basis states $|0\rangle$ and $|1\rangle$ represented by column vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

   (a) Find the density matrix for each of the following pure states:

   $$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

   (b) Find the density matrix for the following mixed state ensembles:

   - $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with probability $\frac{3}{4}$, otherwise $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
   - $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with probability $\frac{1}{2}$, otherwise $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$.

   (c) What can be said in general about the values of $\mathrm{Tr}(\rho)$ and $\mathrm{Tr}(\rho^2)$ for the density matrix ρ of: a pure state; a mixed state?

   Check your statements for one example from part (a) and one from part (b).

3. (a) State the no-cloning theorem.

   (b) Suppose a unitary operator $\hat{U}$ acts as $\hat{U}\,|\Psi_0\rangle = |\Psi\rangle$ and $\hat{U}\,|\Phi_0\rangle = |\Phi\rangle$ where $|\Psi_0\rangle = |\psi\rangle \otimes |0\rangle$, $|\Psi\rangle = |\psi\rangle \otimes |\psi\rangle$, $|\Phi_0\rangle = |\phi\rangle \otimes |0\rangle$, $|\Phi\rangle = |\phi\rangle \otimes |\phi\rangle$, and $|\psi\rangle$, $|\phi\rangle$ and $|0\rangle$ are normalised states.

   By considering suitable inner products show that the above unitary transformations are only possible if $|\psi\rangle$ and $|\phi\rangle$ satisfy certain constraints.

   (c) Find a unitary operator which acts as

$$\hat{U}\Big(\,|n\rangle \otimes |0\rangle\,\Big) = |n\rangle \otimes |n\rangle \ \ , \ n \in \{0, 1\}$$

   on the standard orthonormal basis states of a 2-qubit system.

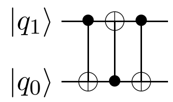
   You can write the operator in Dirac notation or using standard matrix notation.

4. Give a definition of the complexity classes P, NP and EXP.

   For the factoring problem (given an n-bit number x, find its prime factors), a simple algorithm is to consider each prime less than x and check whether it divides x. Which class does this algorithm lie in?

   Show that factoring also lies in the class NP.

5. Consider the quantum circuit



   Determine its action on computational basis states. What is its action on the superposition $H^{\otimes 2}|00\rangle$?

6. (a) Show that the subspace in a 3-qubit Hilbert space spanned by $|001\rangle$ and $|110\rangle$ is a code subspace encoding a single logical qubit which allows for recovery from single bit flip errors.

   (b) Identify the error syndromes diagnosing the three possible bit flips.

SECTION B

7. Consider a single qubit Hilbert space with orthonormal basis states $|0\rangle$ and $|1\rangle$ represented by column vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

   (a) Explain why an arbitrary pure state can be represented by $\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix}$ with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$.

   (b) Show that the density matrix of this state can be written in the form $\rho = \frac{1}{2}(I + r_i\sigma_i)$ where

   $$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   What geometric interpretation of pure qubit states does this give?

   (c) Define mixed states in general, and for a single qubit system, state how mixed states are described geometrically in a similar way to the pure states in part (b). In particular, state (without proof) how pure and mixed states are distinguished in this geometric description.

   (d) Consider a pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ in a general bipartite system. What does it mean to say that $|\Psi\rangle$ is separable or is entangled? How can these two cases be distinguished by considering the reduced density matrix in either subsystem?

   (e) Suppose system A in part (d) is a single qubit system and

   $$|\Psi\rangle = a\,|0\rangle \otimes |\psi\rangle + b\,|1\rangle \otimes |\phi\rangle$$

   where $a, b \in \mathbb{C}$ and $|\psi\rangle$ and $|\phi\rangle$ are normalised.
   Derive the condition for $|\Psi\rangle$ to be normalised and calculate the reduced density matrix for system A.

   (f) What is the entanglement entropy of $|\Psi\rangle$ in the case where:

   - $|\psi\rangle$ is orthogonal to $|\phi\rangle$?
   - $|\psi\rangle = |\phi\rangle$?

8. Consider a single qubit Hilbert space with orthonormal basis states $|0\rangle$ and $|1\rangle$ represented by column vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

   (a) If the system is in the state $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$, what are the possible outcomes, the probability of each outcome, and the final state after each outcome, of a measurement:

   - of $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$?
   - of $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$?

   (b) Now suppose Alice and Bob each have one qubit of a bipartite system in state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right)$.
   If Alice measures σ_3, what are the possible outcomes, the probability of each outcome, and the final state after each outcome?

   (c) Suppose Bob measures σ_3 for his qubit after Alice's measurement. How does his result depend on Alice's? Can Alice and Bob use this to communicate? Justify your answer.

   (d) What are the probabilities of each possible outcome of the measurements on the state $|\beta_{00}\rangle$ if:

   - Alice measures σ_1 and Bob measures σ_3?
   - Alice measures σ_1 and Bob measures σ_1?

   (e) Suppose Alice prepares a large number of states $|\beta_{00}\rangle$ and sends one qubit of each to Bob. Using the results above, outline how Alice and Bob can construct a secret random binary key.

   Describe how Alice and Bob could detect any possible eavesdropping by Eve. In particular, consider the possibility that Eve intercepts the qubits Alice sends to Bob. Eve then decides to measure σ_3 and then forward each qubit to Bob, but Bob cannot tell whether the qubit came from Alice or Eve.

9. Consider the Quantum Fourier Transform, defined as the linear operator $\hat{U}_{FT}$ on an n qubit Hilbert space whose action on basis states $|x\rangle$, $x = 0, \dots 2^n - 1$ is

$$\hat{U}_{FT}|x\rangle = \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n} |y\rangle.$$

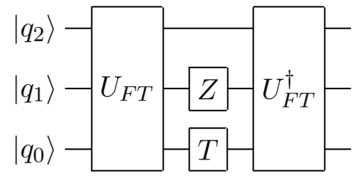
   (a) Show that we can rewrite the transform as a product of states for the individual qubits,

$$\hat{U}_{FT}|x\rangle = \frac{1}{2^{n/2}} \otimes_{l=0}^{n-1} \Big(|0\rangle + \alpha_l |1\rangle \Big),$$

   where you should give a formula for the phases α_l.

   (b) Show directly (that is, without assuming the unitarity of $\hat{U}_{FT}$) that for $x \neq z$, $\hat{U}_{FT}|x\rangle$ is orthogonal to $\hat{U}_{FT}|z\rangle$.

   (c) Consider a 3-qubit system, and consider the unitary transform $\hat{U}_{FT}^{\dagger} T_0 Z_1 \hat{U}_{FT}$, represented by the quantum circuit below.



   Show that this circuit implements the operation $x \to x + 2 \bmod 8$.

10. Consider a function $f(x)$, where x is a 3-bit number, which has two values a_1, a_2 such that $f(a_1) = f(a_2) = 1$, and $f(x) = 0$ for all other values.

   (a) The state

$$|\psi\rangle = H^{\otimes 3}|0\rangle = \frac{1}{\sqrt{8}} \sum_{i=0}^{7} |i\rangle$$

   can be decomposed into a component $|\psi\rangle_a$ in the subspace $\mathcal{H}_a$ spanned by $|a_1\rangle$, $|a_2\rangle$, and a component $|\psi\rangle_\perp$ in the orthogonal subspace $\mathcal{H}_\perp$. Give explicit expressions for the unit normalized vectors

$$|a\rangle = \frac{|\psi\rangle_a}{\sqrt{\langle\psi_a|\psi_a\rangle}}, \quad |\perp\rangle = \frac{|\psi\rangle_\perp}{\sqrt{\langle\psi_\perp|\psi_\perp\rangle}}.$$

   (b) Given a unitary U_f such that

$$U_f|x\rangle \otimes |m\rangle = |x\rangle \otimes |m + f(x)\rangle,$$

   where $|m\rangle$ is the state of a single auxiliary qubit, construct an operation V which reflects vectors in the Hilbert space about the subspace $\mathcal{H}_\perp$. That is, if $|\chi\rangle = |\chi\rangle_a + |\chi\rangle_\perp$ with $|\chi\rangle_a \in \mathcal{H}_a$ and $|\chi\rangle_\perp \in \mathcal{H}_\perp$,

$$V|\chi\rangle = -|\chi\rangle_a + |\chi\rangle_\perp.$$

   (c) Show that if we have a vector in the two-dimensional subspace spanned by $|a\rangle$ and $|\perp\rangle$, applying V and

$$D = 2|\psi\rangle\langle\psi| - I$$

   rotates the state in this subspace, and find the rotation angle.

   (d) Give an algorithm to use this rotation to find one of the special values a_1, a_2.