Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH4051-WE01

Title:

General Relativity IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. Let $x^\mu = (x, y)$ denote Cartesian coordinates on the plane, and let $\tilde{x}^\mu = (r, \phi)$ denote polar coordinates, where $x = r\cos\phi$ and $y = r\sin\phi$.

   (a) If the covector field W has components $W_\mu = (-y, x)$ in Cartesian coordinates, compute its components $\widetilde{W}_\mu$ in polar coordinates.

   (b) If the vector field V has components $\widetilde{V}^\mu = (1, 0)$ in polar coordinates, compute its components V^μ in Cartesian coordinates.

2. Consider the metric $ds^2 = x\,dt^2 - 2\,dt\,dx - x\,dx^2 - dy^2 - dz^2$, and the curve given by $t(s) = -s^2/4$, $x(s) = s$, $y = z = 0$ for $s \in (0, 1)$.

   (a) What is the signature of this metric?

   (b) Compute the quantity $\Delta = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, where $\dot{x}^\mu = dx^\mu/ds$. Is the curve timelike, null or spacelike?

   (c) Compute the proper length D of the curve.

3. State the Bianchi identity, and use it to show that $\nabla^\mu R_{\mu\nu} = k\nabla_\nu R$, where k is a constant which you should determine. Hence show that if $R_{\mu\nu} = f g_{\mu\nu}$ in a four-dimensional space-time, then the scalar f has to be constant.

4. The action of a scalar field ϕ is

$$S[\phi] = \int d^4x\,\sqrt{-g}\,\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right),$$

   where $V(\phi)$ is some function.

   (a) Compute the equation of motion of the scalar field.

   (b) Show that the equation of motion implies $\nabla_\mu T^{\mu\nu} = 0$, where

$$T_{\mu\nu} = 2(\nabla_\mu\phi)(\nabla_\nu\phi) - g_{\mu\nu}[(\nabla_\alpha\phi)(\nabla^\alpha\phi) - V(\phi)].$$

5. The action of a particle with mass m and electromagnetic charge q is

$$S = \int ds \left(-m\sqrt{g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} + qA_\mu \dot{X}^\mu \right),$$

where A_μ is the vector potential.

(a) Show that the action is invariant under reparametrization $s \to s'(s)$.

(b) Find the modified geodesic equation obeyed by the charged particle. (You may use known results for the variation of the term in the action containing m.)

6. In a matter-dominated FRW universe with $\kappa = 1$ and $\Lambda = 0$, the solution to the Friedmann equation can be written in the parametric form

$$a(\eta) = \frac{a_0}{2}(1-\cos\eta), \quad t(\eta) = \frac{a_0}{2}(\eta - \sin\eta),$$

where a_0 is a constant (which characterizes the maximum size of the universe). A photon is emitted at the big bang. By writing the positive curvature spatial slice as a three-sphere $d\chi^2 + \sin^2\chi d\Omega^2$, and using η as a time coordinate, show that the photon travels all the way around the universe by the time of the end of the universe i.e. back to the point on the 3-sphere where it started.

SECTION B

7. Consider the two-dimensional space-time with local coordinates $(x^0, x^1) = (t, r)$ and metric $ds^2 = r^{-2}(dt^2 - dr^2)$, where $r > 0$.

(a) Find the geodesic $r = r(t)$ satisfying $r = 1$ and $dr/dt = 0$ at $t = 0$.

(b) Compute the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$.

(c) Find the vector field V^μ which is covariantly-constant along the curve $r = 2$, with $V^\mu = (1, 0)$ at $(t, r) = (0, 2)$.

8. (i) A two-dimensional space-time has the metric $ds^2 = x^2\, dt^2 - dx^2$. Write out the geodesic equations, and show that the quantity $P = \mathrm{e}^t(x\dot{t} + k\dot{x})$ is constant along affinely-parametrized geodesics with tangent vector $V^\mu = (\dot{t}, \dot{x})$, for some constant k which you should determine.

   (ii) Let $x^\mu(u, s)$ denote a one-parameter family of geodesics, with s being an affine parameter along each geodesic. Define $V^\mu = \partial x^\mu/\partial s$ and $U^\mu = \partial x^\mu/\partial u$. Derive the equation of geodesic deviation, expressing $A^\alpha = V^\mu \nabla_\mu (V^\nu \nabla_\nu U^\alpha)$ in terms of V^μ, U^μ and the Riemann tensor. You may assume without proof that $V^\mu \nabla_\mu U^\nu = U^\mu \nabla_\mu V^\nu$.

   (iii) If $P_{\mu\nu}$ is a tensor field, show that

$$g^{\mu\beta}[\nabla_\alpha, \nabla_\beta] P_{\mu\nu} = A g^{\mu\beta} R_{\alpha\beta} P_{\mu\nu} + B R_{\alpha\beta\nu\sigma} P^{\beta\sigma},$$

   where A and B are constants which you should determine.

9. Consider the metric produced by a mass distribution in three space-time dimensions (a 3d "star"). We use a rotationally-symmetric metric of the form

$$ds^2 = e^{2A(r)} dt^2 - e^{2B(r)} dr^2 - r^2 d\theta^2,$$

where θ is an angular coordinate with periodicity 2π. The nonzero components of the Einstein tensor are

$$G_{tt} = e^{2(A-B)} B'/r, \quad G_{rr} = A'/r, \quad G_{\theta\theta} = e^{-2B} r^2 \left(A'^2 - A'B' + A''\right).$$

We study the Einstein equation sourced by a matter distribution that is pressureless dust at rest at radii $r < R$, so

$$T_{\mu\nu} = \rho u_\mu u_\nu, \quad u^\mu = e^{-A(r)} \delta^\mu_t \quad \text{for } r < R,$$

and is the vacuum $T_{\mu\nu} = 0$ for $r > R$. R defines the edge of the star and is given.

   (a) Show that $A(r)$ is a constant, and argue that it can be set to *zero*. Find the most general solution for $B(r)$ when $r < R$.

   (b) Find a solution to the metric for $r > R$. Write down a metric in the (t, r, θ) coordinates that solves Einstein's equations everywhere, and is smooth both at the origin at $r = 0$ and across $r = R$. Your answer should have no integration constants remaining.

   (c) Find a coordinate transformation to new coordinates (τ, σ, ϕ) to put the metric for $r > R$ in the form $ds^2 = d\tau^2 - d\sigma^2 - \sigma^2 d\phi^2$. In terms of ρ, R, and G, what is the periodicity of the new angular coordinate ϕ? Describe in words the geometry inside and outside the "star".

10. Consider the space-time with metric $ds^2 = r^2 d\eta^2 - dr^2$, where $r > 0$.

   (a) Write the geodesic equation for massive particles in the form

   $$\frac{1}{2}\left(\frac{dr}{ds}\right)^2 + V(r) = E,$$

   and describe in words the resulting trajectories.

   (b) Consider the coordinate transformation to the coordinates (t, x) given by

   $$x = r\cosh\eta, \quad t = r\sinh\eta,$$

   whose inverse map is

   $$\eta = \tanh^{-1}\left(\frac{t}{x}\right), \quad r = \sqrt{x^2 - t^2}\ .$$

   Compute the metric in the (t, x) coordinate system. (Hint: $\frac{d}{dx}\tanh^{-1}(x) = 1/(1 - x^2)$).

   (c) What part of the space labeled by (t, x) is covered by the (η, r) coordinates? Draw a picture illustrating this, and indicate $r = 0$ on the picture. Interpret the trajectories found in part (a) from this point of view.

   (d) Compute the magnitude of the acceleration vector $a^\nu \equiv u^\mu \nabla_\mu u^\nu$ of a particle with worldline $r(s) = r_0$, $\eta(s) = s/r_0$, where s is an affine parameter (*i.e.* it is at rest at constant $r = r_0$ in the (η, r) coordinates.)