Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH4061-WE01

Title:
Advanced Quantum Theory IV

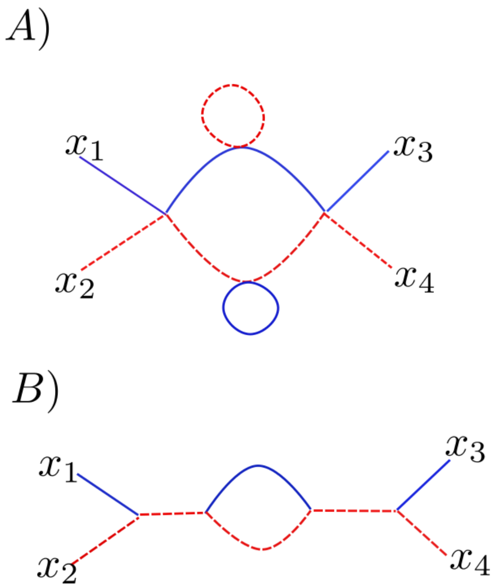
Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

SECTION A

1. A scalar field $\phi(x)$ of mass m and another scalar field $\Phi(x)$ of mass M both feature in two different interacting scalar field theories, with (non-derivative) interaction couplings. The graphs below show one particular graph for theory A) and one for theory B).



   (a) Based on your general knowledge about Feynman graphs in position space, write down the expression for graph A) as integral in position space. Call the interaction coupling λ. Also write down the interaction term in the action for this theory, with an appropriate normalisation factor to match your expression for the Feynman graph.

   (b) Write down the expression for graph B) in momentum space. Call the interaction coupling η. Apply the LSZ reduction formula to this expression, to modify it so that it gives one of the terms in the scattering amplitude of external particles.

   Note: you do not need to do any of the integrals.

2. The fields $\hat{\varphi}_i(x)$ with $i = 1, 2, 3$ are three free scalar fields.

   (a) Give the result of Wick's theorem when applied to the expression

$$T\left(\hat{\varphi}_1^2(x)\hat{\varphi}_2(y_1)\hat{\varphi}_3^2(w)\hat{\varphi}_2(y_2)\right). \qquad (1)$$

   Explain all the symbols which appear in the result. Use your result to evaluate the vacuum expectation value of (1). If any terms vanish, explain why.

   (b) Evaluate the expression

$$\langle 0|T\left(\hat{\varphi}_1(x_1)\hat{\varphi}_1(x_2)\cdots\hat{\varphi}_1(x_{101})\hat{\varphi}_3(w_1)\hat{\varphi}_2^2(y_1)\hat{\varphi}_3(w_2)\right)|0\rangle\,.$$

   Explain your answer.

3. (a) For the free complex scalar field, write down the definition of the Feynman propagator. Explain the physical motivation for this formula.

   (b) Show that

$$\int \mathrm{d}^4 z\, G(x-z)G(z-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{e}^{ik\cdot(x-y)} \left(\frac{i\hbar}{-k\cdot k - m^2 + i\epsilon} \right)^2 ,$$

   where $G(x-y)$ is the Feyman propagator for the scalar field.

4. Consider the integral

$$Z_\lambda = \int_{-\infty}^{\infty} e^{-\phi^2 - \lambda\phi^4} \mathrm{d}\phi \,. \tag{2}$$

   (a) The 'generating function' $Z_\lambda[J]$ is defined to be

$$Z_\lambda[J] = \int_{-\infty}^{\infty} e^{-\phi^2 - \lambda\phi^4 + J\phi} \mathrm{d}\phi \,. \tag{3}$$

   Show that this can be rewritten as

$$Z_\lambda[J] = \exp\left[-\lambda \frac{\mathrm{d}^4}{\mathrm{d}J^4} \right] \int_{-\infty}^{\infty} e^{-\phi^2 + J\phi} \mathrm{d}\phi \,. \tag{4}$$

   (b) Evaluate the integral in (4) by completing the square in the exponent and calculating the Gaussian integral.

   (c) Now compute the zeroth and first order terms in the expansion of $Z_\lambda[J]$ in powers of λ. Show that the first two terms of the series expansion of $Z_\lambda[J=0]$ near $\lambda = 0$ are given by

$$Z_\lambda[J=0] = \sqrt{\pi} - \frac{3\sqrt{\pi}}{4}\lambda + \mathcal{O}(\lambda^2) \,. \tag{5}$$

5. Consider the action for a p-brane, that is, an extended object with p space-like directions and one time-like direction. The Polyakov action for this object reads

$$S = \frac{1}{2} \int \Big(-\sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + (p-1)\sqrt{-h} \Big) \,\mathrm{d}^{p+1}\sigma \,. \tag{6}$$

Here we have collectively denoted the $p+1$ world-volume directions by σ. The tensor $h_{\alpha\beta}$ is the world-volume metric and h is its determinant. $X^\mu(\sigma)$ are the embedding coordinates.

   (a) Compute the equation of motion for $h^{\alpha\beta}$. You may assume that $p \neq 1$.

   (b) Use this equation to find $h_{\alpha\beta}$ in terms of the derivatives of the embedding coordinates.

   (c) Insert the solution back into the action to find the Nambu-Goto action for the p-brane.

6. The four-point scattering amplitude for a real scalar field with ϕ^4 interaction can be written in terms of the Mandelstam variables s, t and u as

$$\Gamma^{(4)} = \frac{-i}{\hbar}\lambda + iC\lambda^2 \left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right)\right] + \mathcal{O}(\lambda^3)\,, \qquad (7)$$

where λ is the coupling constant and C is some positive number.

(a) Where did the terms proportional to λ^2 come from? What is the meaning of Λ and why did we introduce it?

(b) The expression above depends on Λ, but that is only superficially so. Why? Perform the steps necessary to express $\Gamma^{(4)}$ in terms of physically measurable quantities.

SECTION B

7. An interacting, real scalar field theory has the action

$$S = \int \mathrm{d}^4x \left(-\frac{1}{2}(\partial_\mu \varphi(x))(\partial^\mu \varphi(x)) - \frac{1}{2}m^2\varphi(x)^2 + \frac{\lambda}{4!}\varphi(x)^4 \right)$$

   (a) Write down the general formula which expresses the generic two-point correlator in the interacting theory $\langle \Omega | \hat{\varphi}(x_1)\hat{\varphi}(x_2)|\Omega\rangle$ in terms of the fields in the free theory.

   (b) Evaluate the numerator in the formula which you have given in a) to order λ^2. Express your results in terms of expressions with Feynman propagators.

   (c) Evaluate the denominator in the formula which you have given in b), up to and including λ^2 order. Express your results in terms of expressions with Feynman propagators.

   (d) By combining the results from the two previous parts, show explicitly that all the bubble diagrams get removed.

   Hint: First answer this purely graphically (i.e. do not write Feynman diagrams as mathematical expressions) and only at the end write down what the diagrams correspond to mathematically.

8. The action for two scalar fields $\varphi_1(x)$ and $\varphi_2(x)$ is given by

$$S = \int \mathrm{d}^4x \Big[-\frac{1}{2}\left((\partial_\mu\varphi_1)^2 + (\partial_\mu\varphi_2)^2 \right) - \frac{1}{2}m^2\left(\varphi_1^2 + \varphi_2^2\right) + \lambda\left((\partial_\mu\varphi_1)^2 + (\partial_\mu\varphi_2)^2 \right)^2 \Big] .$$

   (a) Derive the equations of motion for this action.

   (b) Show that this action is invariant under the infinitesimal transformations

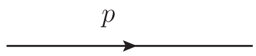
$$\varphi_1(x) \to \cos(\alpha)\,\varphi_1(x) + \sin(\alpha)\,\varphi_2(x)\,,$$
$$\varphi_2(x) \to -\sin(\alpha)\,\varphi_1(x) + \cos(\alpha)\,\varphi_2(x)\,,$$

   where α is an arbitrary constant parameter.

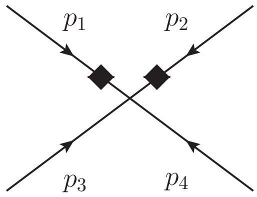
   (c) Using the Noether procedure derive the conserved current and charge associated to the symmetry given above.

   (d) Now assume that $\lambda = 0$. Starting from the classical expression for the current which you have derived in c), derive a quantum form expressed in terms of normal-ordered creation and annihilation operators.

9. Consider the following set of Feynman rules for a real scalar field,



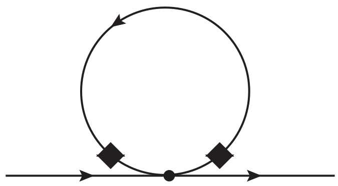
$$\frac{-i\hbar}{p^2 + m^2 - i\epsilon}$$



$$-\frac{i}{\hbar}\lambda (p_1)^\mu (p_2)_\mu$$

(a) Write down the action which would lead to this set of Feynman rules.

(b) Consider the Feynman diagram



Write down the mathematical expression for the amplitude corresponding to this diagram.

(c) Give the Taylor expansion of this expression to third order in m^2 (i.e. to order m^6). Now assume that your loop momentum is Euclidean, and integrate each of these terms with a loop momentum cutoff Λ and determine the behaviour for large Λ. You may assume that Taylor expansion and integration can be exchanged.

(d) Explain in words what needs to be done in order to make physical sense of the result you obtained above.

10. The stress tensor for the classical bosonic string is given by

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu \,, \tag{8}$$

where μ is a space-time index and the other Greek indices take values in the world-volume directions $\{\tau, \sigma\}$.

(a) Explain why, when evaluated on a physical solution, the stress tensor needs to vanish.

(b) Determine the form of the stress tensor in the special case when the world-sheet metric is flat,

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\beta} .$$

Express your answer in terms of the left-moving and right-moving components X_L^μ and X_R^μ of the world-sheet embedding coordinates.

(c) The left-moving and right-moving components can be expanded in modes as

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{4\pi T}(\tau - \sigma)\, p^\mu + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau - \sigma)} \,, \tag{9}$$

and similar for X_L^μ, where T is the string tension. Use this to express $\dot{X}_R^2$ and $\dot{X}_L^2$ in the form

$$\begin{aligned} \dot{X}_R^2 &= 2 \sum_{m=-\infty}^{\infty} L_m e^{-im(\tau-\sigma)} \,, \\ \dot{X}_L^2 &= 2 \sum_{m=-\infty}^{\infty} \tilde{L}_m e^{-im(\tau+\sigma)} \,. \end{aligned} \tag{10}$$

Determine the L_m and $\tilde{L}_m$ in terms of $\alpha_0^\mu = p^\mu / \sqrt{4\pi T}$ and α_n^μ.

(d) Compute the components T_{--}, T_{++} and T_{+-} by transforming according to

$$T_{--} = \frac{\partial \sigma^\alpha}{\partial \sigma^-} \frac{\partial \sigma^\beta}{\partial \sigma^-} T_{\alpha\beta}$$

and so on, where

$$\sigma^\pm = \tau \pm \sigma \,.$$

Determine the components T_{--}, T_{++} and T_{+-} in terms of the L_m and $\tilde{L}_m$.