Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH4131-WE01

Title: Probability IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

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SECTION A

1. An *n-step path* $(s_0, s_1, \ldots, s_n)$ of integers satisfies $|s_{k+1} - s_k| = 1$ for $0 \le k \le n-1$. For $x, y \in \mathbb{Z}$, let $N_n(x, y)$ denote the number of n-step paths with $s_0 = x$ and $s_n = y$.

   (i) If $a, b > 0$, show that the number of n-step paths with $s_0 = a$ and $s_n = b$ that visit 0 is $N_n(-a, b)$.

   (ii) If $a, b > 0$, show that the number of n-step paths such that $s_0 = 0$, $s_1 > -a$, $s_2 > -a$, $\ldots$, $s_{n-1} > -a$, $s_n = b$ is $N_n(0, b) - N_n(0, 2a + b)$.

   (iii) If $a > b > 0$, show that the number of n-step paths such that $s_0 = 0$, $s_1 < a$, $s_2 < a$, $\ldots$, $s_{n-1} < a$, $s_n = b$ is $N_n(0, b) - N_n(0, 2a - b)$.

2. Let $S_0, S_1, \ldots$ be simple random walk with parameter $p \in (0, 1)$ started at $S_0 = 0$. Let $u_{2n} := \mathbb{P}(S_{2n} = 0)$ and set $q := 1 - p$.

   (a) In the Taylor expansion

   $$(1 + x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n,$$

   explain the definition of the extended binomial coefficient $\binom{-1/2}{n}$.

   (b) Show that $u_{2n} = \binom{2n}{n} p^n q^n = \binom{-1/2}{n} (-4pq)^n$ (prove both equalities).

   (c) Hence obtain a formula for the generating function $\mathcal{U}(s) := \sum_{n \ge 0} u_{2n} s^{2n}$.

   (d) Deduce a formula for $\mathcal{F}(s) := \sum_{k \ge 1} f_{2k} s^{2k}$, where f_{2k} is the probability that the walk returns to the origin for the first time at time $2k$.

   (e) Hence show that the probability that the walk ever returns to zero is $1 - |p - q|$.

3. (i) State Jensen's inequality. Show that $-\log x$ is a convex function on $(0, \infty)$. Deduce that $n^{-1} \sum_{i=1}^{n} x_i \ge (\prod_{i=1}^{n} x_i)^{1/n}$ for any positive integer n and any positive real numbers $x_1, \ldots, x_n$.

   (ii) Let X be a random variable with characteristic function φ_X. Show that φ_X is real-valued if and only if X is symmetric (i.e., X and $-X$ have the same distribution).

   (iii) Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with mean zero and finite variance, and let $S_n = \sum_{k=1}^{n} \xi_k$. Suppose that

   $$0 < \lim_{n \to \infty} \mathbb{P}\left(|n^{-1/2} S_n| < a\right) = b < 1.$$

   Find an expression for $\mathbb{V}\mathrm{ar}(\xi_1)$ in terms of a, b, and Φ^{-1}, the inverse of the standard normal distribution function.

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4. For a fixed constant $\lambda > 0$, let X_1 and X_2 be independent $\mathsf{Exp}(\lambda)$ random variables, and let $X_{(1)}$ and $X_{(2)}$ be the corresponding order statistics.

   (a) Show that $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent and find their distributions.

   (b) Compute $\mathbb{E}(X_{(2)} \mid X_{(1)} = x_1)$ and $\mathbb{E}(X_{(1)} \mid X_{(2)} = x_2)$.

5. Carefully define the stochastic order $\preccurlyeq$ for random variables. In the questions below, justify your answer by proving the result or giving a counter-example:

   (i) Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ be Gaussian random variables.

      (a) If $\mu_X \le \mu_Y$ but $\sigma_X^2 = \sigma_Y^2$, is it true that $X \preccurlyeq Y$?

      (b) If $\mu_X = \mu_Y$ but $\sigma_X^2 \le \sigma_Y^2$, is it true that $X \preccurlyeq Y$?

   (ii) Let X be a Binomial $\mathsf{Bin}(k, p)$ random variable and let Y be a Binomial $\mathsf{Bin}(m, p)$ random variable, where $k \le m$. Is it true that $X \preccurlyeq Y$?

   (iii) Let X be a geometric $\mathsf{Geom}(p)$ random variable with parameter $p \in (0, 1)$, ie., $\mathbb{P}(X > k) = (1 - p)^k$ for integer $k \ge 0$. Let, further, Y be geometric with parameter r, $p \le r \le 1$. Are variables X and Y stochastically ordered?

   In your answer you should clearly state and carefully apply any result you use.

6. Let r balls be placed randomly into n boxes. Denote by $N = N_{r,n}$ the number of empty boxes, and let $p_k(r, n) = \mathbb{P}(N = k)$ be the probability that exactly k boxes are empty.

   (a) Show that for each $k \ge 0$,
   $$p_k(r, n) = \binom{n}{k} \left(1 - \frac{k}{n}\right)^r p_0(r, n - k) \,.$$

   (b) Suppose that for some constant $\lambda > 0$, we have $ne^{-r/n} \to \lambda$ as $n \to \infty$. Use the equality in part (a) to show that
   $$p_k(r, n) \to \frac{\lambda^k}{k!} e^{-\lambda}$$
   for all $k \ge 0$, equivalently, that in the specified limit the distribution of N converges to $\mathsf{Poi}(\lambda)$, the Poisson distribution with parameter λ.

   In your answer you can use without proof the inequality $|x + \log(1 - x)| \le x^2$ with real $|x| \le 1/2$.

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SECTION B

7. Consider a sequence of Bernoulli trials with success probability $p \in (0, 1)$. Fix a positive integer r and let $\mathcal{E}$ denote the event that a run of r successes is observed; we do not allow overlapping runs, so that $\mathcal{E}$ is a recurrent event. Let u_n be the probability that $\mathcal{E}$ occurs on the nth trial.

   (a) By considering the event that trials n, $n-1$, $\ldots$, $n-r$ all result in success, show that
   $$u_n + u_{n-1}p + \cdots + u_{n-r+1}p^{r-1} = p^r, \text{ for } n \geq r.$$
   Also find $u_0, u_1, \ldots, u_{r-1}$.

   (b) Multiply both sides of the equation in part (a) by s^n and sum over $n \geq r$ to obtain an expression involving the generating function $\mathcal{U}(s) := \sum_{n \geq 0} u_n s^n$. Hence find $\mathcal{U}(s)$ in terms of s, p, $q := 1 - p$, and r.

   (c) Deduce an expression for $\mathcal{F}(s) := \sum_{k \geq 1} f_k s^k$, where f_k is the probability of observing $\mathcal{E}$ for the *first* time on the kth trial.

   Use your expression for $\mathcal{F}(s)$ from part (c) to answer the following.

   (d) Show that $\mathcal{E}$ occurs eventually with probability 1.

   (e) Find $\mu := \sum_{k \geq 1} k f_k$ in terms of p, q, and r.

8. (a) State the definition of *convergence in probability* ($\xrightarrow{p}$).

   Given a sequence of random variables $X_1, X_2, \ldots$ we say X_n is *uniformly bounded* if there exists a constant $M < \infty$ such that $\mathbb{P}(|X_n| \leq M) = 1$ for all n.

   (b) Show that if X_n is uniformly bounded and $Y_n \xrightarrow{p} 0$, then $X_n Y_n \xrightarrow{p} 0$.

   (c) Show that if $X_n \xrightarrow{p} X$ and X_n is uniformly bounded with bound M, then $\mathbb{P}(|X| \leq M) = 1$.

   (d) Suppose that $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$. If X_n and Y_n are uniformly bounded, show that $X_n Y_n \xrightarrow{p} XY$.
   Hint: Write $X_n Y_n - XY = X_n(Y_n - Y) + Y(X_n - X)$.

   For the final two parts of the question, X_n and Y_n are *not* assumed to be uniformly bounded.

   (e) Show that if $X_n \xrightarrow{p} X$ for a finite random variable X then for any $\delta > 0$ there exists a constant $M < \infty$ such that $\mathbb{P}(|X_n| > M) \leq \delta$ for all n sufficiently large.

   (f) Suppose that $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$. Show that $X_n Y_n \xrightarrow{p} XY$.

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9. (a) Carefully define order statistics for a sample of independent identically distributed random variables.

Let variables $X_1, \ldots, X_n$ be independent with common cumulative distribution function (c.d.f.) $F(\,\cdot\,)$. Write $F_{X_{(k)}}(x) = \mathbb{P}(X_{(k)} \le x)$ for the c.d.f. of the kth order variable.

(b) Show that $F_{X_{(n)}}(x) = \big(F(x)\big)^n$ and $F_{X_{(1)}}(x) = 1 - \big(1 - F(x)\big)^n$.

(c) Show that for $k = 1, \ldots, n$,

$$F_{X_{(k)}}(x) = \sum_{\ell=k}^{n} \binom{n}{\ell} \big(F(x)\big)^{\ell} \big(1 - F(x)\big)^{n-\ell} .$$

(d) Show that for $k = 1, \ldots, n$,

$$F_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \int_0^{F(x)} y^{k-1}(1-y)^{n-k}\, dy .$$

(e) Now suppose in addition that $X_k \sim \mathcal{U}(0,1)$, the uniform distribution on $(0,1)$. Find the probability density function $f_{X_{(k)}}(x)$ of $X_{(k)}$ and compute the expectation $\mathbb{E}X_{(k)}$.

10. Consider bond percolation on the square lattice $\mathbb{Z}^2$, with every bond independently open with probability $p \in [0,1]$.

(a) Carefully define the percolation probability $\theta(p)$ and show that it is a non-decreasing function of p; hence define the critical value p_{c}.

(b) Show that $\theta(p) = 0$ for $p > 0$ small enough; hence deduce that $p_{\mathsf{c}} \ge p'$ for some $p' > 0$.

(c) Show that $\theta(p) > 0$ for $1 - p > 0$ small enough; hence deduce that $p_{\mathsf{c}} \le p''$ for some $p'' < 1$.

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SECTION C

11. (a) Carefully state Cramér's theorem on large deviations for sums of i.i.d. random variables.

(b) Show that under the conditions of Cramér's theorem the rate function I satisfies the inequality $I(a) \geq 0$ for all $a \in \mathbb{R}$; for which value(s) of a does equality hold?

Let $X_1, X_2, \ldots$ be independent Poisson random variables with parameter $\lambda > 0$, and let $S_n = X_1 + \cdots + X_n$.

(c) Compute the moment generating function

$$\varphi(t) := \mathbb{E}\exp\{tX_1\}$$

and the corresponding rate function $I(a)$.

(d) Verify in this example that $I''(\mathbb{E}X_1) = 1/\mathbb{V}\mathrm{ar}X_1$.

(e) Find the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq an)$$

for $a > \mathbb{E}X_1$. What happens when $a \leq \mathbb{E}X_1$?

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