

EXAMINATION PAPER

Examination Session: May

2017

Year:

Exam Code:

MATH4131-WE01

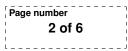
Title:

Probability IV

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted:		
		Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for: the best TWO answers from Section the best THREE answers from Section AND the answer to the question in Section B and C carry T those in Section A.	on B, ection C.	any marks as
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Revision:



SECTION A

- 1. An *n*-step path (s_0, s_1, \ldots, s_n) of integers satisfies $|s_{k+1} s_k| = 1$ for $0 \le k \le n-1$. For $x, y \in \mathbb{Z}$, let $N_n(x, y)$ denote the number of *n*-step paths with $s_0 = x$ and $s_n = y$.
 - (i) If a, b > 0, show that the number of *n*-step paths with $s_0 = a$ and $s_n = b$ that visit 0 is $N_n(-a, b)$.
 - (ii) If a, b > 0, show that the number of *n*-step paths such that $s_0 = 0, s_1 > -a, s_2 > -a, \ldots, s_{n-1} > -a, s_n = b$ is $N_n(0, b) N_n(0, 2a + b)$.
 - (iii) If a > b > 0, show that the number of *n*-step paths such that $s_0 = 0$, $s_1 < a$, $s_2 < a$, ..., $s_{n-1} < a$, $s_n = b$ is $N_n(0, b) N_n(0, 2a b)$.
- 2. Let S_0, S_1, \ldots be simple random walk with parameter $p \in (0, 1)$ started at $S_0 = 0$. Let $u_{2n} := \mathbb{P}(S_{2n} = 0)$ and set q := 1 - p.
 - (a) In the Taylor expansion

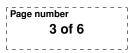
$$(1+x)^{-1/2} = \sum_{n=0}^{\infty} {\binom{-1/2}{n}} x^n,$$

explain the definition of the extended binomial coefficient $\binom{-1/2}{n}$.

- (b) Show that $u_{2n} = \binom{2n}{n} p^n q^n = \binom{-1/2}{n} (-4pq)^n$ (prove both equalities).
- (c) Hence obtain a formula for the generating function $\mathcal{U}(s) := \sum_{n>0} u_{2n} s^{2n}$.
- (d) Deduce a formula for $\mathcal{F}(s) := \sum_{k \ge 1} f_{2k} s^{2k}$, where f_{2k} is the probability that the walk returns to the origin for the first time at time 2k.
- (e) Hence show that the probability that the walk ever returns to zero is 1 |p q|.
- 3. (i) State Jensen's inequality. Show that $-\log x$ is a convex function on $(0, \infty)$. Deduce that $n^{-1} \sum_{i=1}^{n} x_i \ge (\prod_{i=1}^{n} x_i)^{1/n}$ for any positive integer n and any positive real numbers x_1, \ldots, x_n .
 - (ii) Let X be a random variable with characteristic function φ_X . Show that φ_X is real-valued if and only if X is symmetric (i.e., X and -X have the same distribution).
 - (iii) Let ξ_1, ξ_2, \ldots be i.i.d. random variables with mean zero and finite variance, and let $S_n = \sum_{k=1}^n \xi_k$. Suppose that

$$0 < \lim_{n \to \infty} \mathbb{P}\left(|n^{-1/2} S_n| < a \right) = b < 1.$$

Find an expression for $\mathbb{V}ar(\xi_1)$ in terms of a, b, and Φ^{-1} , the inverse of the standard normal distribution function.



- 4. For a fixed constant $\lambda > 0$, let X_1 and X_2 be independent $\mathsf{Exp}(\lambda)$ random variables, and let $X_{(1)}$ and $X_{(2)}$ be the corresponding order statistics.
 - (a) Show that $X_{(1)}$ and $X_{(2)} X_{(1)}$ are independent and find their distributions.
 - (b) Compute $\mathbb{E}(X_{(2)} | X_{(1)} = x_1)$ and $\mathbb{E}(X_{(1)} | X_{(2)} = x_2)$.
- 5. Carefully define the stochastic order \preccurlyeq for random variables. In the questions below, justify your answer by proving the result or giving a counter-example:
 - (i) Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ be Gaussian random variables.
 - (a) If $\mu_X \leq \mu_Y$ but $\sigma_X^2 = \sigma_Y^2$, is it true that $X \preccurlyeq Y$?
 - (b) If $\mu_X = \mu_Y$ but $\sigma_X^2 \leq \sigma_Y^2$, is it true that $X \preccurlyeq Y$?
 - (ii) Let X be a Binomial Bin(k, p) random variable and let Y be a Binomial Bin(m, p) random variable, where $k \leq m$. Is it true that $X \preccurlyeq Y$?
 - (iii) Let X be a geometric Geom(p) random variable with parameter $p \in (0, 1)$, ie., $\mathbb{P}(X > k) = (1 - p)^k$ for integer $k \ge 0$. Let, further, Y be geometric with parameter $r, p \le r \le 1$. Are variables X and Y stochastically ordered?

In your answer you should clearly state and carefully apply any result you use.

- 6. Let r balls be placed randomly into n boxes. Denote by $N = N_{r,n}$ the number of empty boxes, and let $p_k(r,n) = \mathbb{P}(N=k)$ be the probability that exactly k boxes are empty.
 - (a) Show that for each $k \ge 0$,

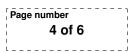
$$p_k(r,n) = \binom{n}{k} \left(1 - \frac{k}{n}\right)^r p_0(r,n-k) \,.$$

(b) Suppose that for some constant $\lambda > 0$, we have $ne^{-r/n} \to \lambda$ as $n \to \infty$. Use the equality in part (a) to show that

$$p_k(r,n) \to \frac{\lambda^k}{k!} e^{-\lambda}$$

for all $k \ge 0$, equivalently, that in the specified limit the distribution of N converges to $\mathsf{Poi}(\lambda)$, the Poisson distribution with parameter λ .

In your answer you can use without proof the inequality $|x + \log(1 - x)| \le x^2$ with real $|x| \le 1/2$.



SECTION B

- 7. Consider a sequence of Bernoulli trials with success probability $p \in (0, 1)$. Fix a positive integer r and let \mathcal{E} denote the event that a run of r successes is observed; we do not allow overlapping runs, so that \mathcal{E} is a recurrent event. Let u_n be the probability that \mathcal{E} occurs on the *n*th trial.
 - (a) By considering the event that trials $n, n-1, \ldots, n-r$ all result in success, show that

$$u_n + u_{n-1}p + \dots + u_{n-r+1}p^{r-1} = p^r$$
, for $n \ge r$.

Also find $u_0, u_1, ..., u_{r-1}$.

- (b) Multiply both sides of the equation in part (a) by s^n and sum over $n \ge r$ to obtain an expression involving the generating function $\mathcal{U}(s) := \sum_{n\ge 0} u_n s^n$. Hence find $\mathcal{U}(s)$ in terms of s, p, q := 1 - p, and r.
- (c) Deduce an expression for $\mathcal{F}(s) := \sum_{k \ge 1} f_k s^k$, where f_k is the probability of observing \mathcal{E} for the *first* time on the *k*th trial.

Use your expression for $\mathcal{F}(s)$ from part (c) to answer the following.

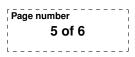
- (d) Show that \mathcal{E} occurs eventually with probability 1.
- (e) Find $\mu := \sum_{k>1} k f_k$ in terms of p, q, and r.
- 8. (a) State the definition of *convergence in probability* (\xrightarrow{p}) .

Given a sequence of random variables X_1, X_2, \ldots we say X_n is uniformly bounded if there exists a constant $M < \infty$ such that $\mathbb{P}(|X_n| \leq M) = 1$ for all n.

- (b) Show that if X_n is uniformly bounded and $Y_n \xrightarrow{p} 0$, then $X_n Y_n \xrightarrow{p} 0$.
- (c) Show that if $X_n \xrightarrow{p} X$ and X_n is uniformly bounded with bound M, then $\mathbb{P}(|X| \leq M) = 1$.
- (d) Suppose that $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$. If X_n and Y_n are uniformly bounded, show that $X_n Y_n \xrightarrow{p} XY$. *Hint:* Write $X_n Y_n - XY = X_n (Y_n - Y) + Y (X_n - X)$.

For the final two parts of the question, X_n and Y_n are *not* assumed to be uniformly bounded.

- (e) Show that if $X_n \xrightarrow{p} X$ for a finite random variable X then for any $\delta > 0$ there exists a constant $M < \infty$ such that $\mathbb{P}(|X_n| > M) \leq \delta$ for all n sufficiently large.
- (f) Suppose that $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$. Show that $X_n Y_n \xrightarrow{p} XY$.



- Exam code MATH4131-WE01
- 9. (a) Carefully define order statistics for a sample of independent identically distributed random variables.

Let variables X_1, \ldots, X_n be independent with common cumulative distribution function (c.d.f.) $F(\cdot)$. Write $F_{X_{(k)}}(x) = \mathbb{P}(X_{(k)} \leq x)$ for the c.d.f. of the kth order variable.

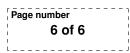
- (b) Show that $F_{X_{(n)}}(x) = (F(x))^n$ and $F_{X_{(1)}}(x) = 1 (1 F(x))^n$.
- (c) Show that for $k = 1, \ldots, n$,

$$F_{X_{(k)}}(x) = \sum_{\ell=k}^{n} \binom{n}{\ell} (F(x))^{\ell} (1 - F(x))^{n-\ell}.$$

(d) Show that for $k = 1, \ldots, n$,

$$F_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} \int_0^{F(x)} y^{k-1} (1-y)^{n-k} \, dy \, .$$

- (e) Now suppose in addition that $X_k \sim \mathcal{U}(0, 1)$, the uniform distribution on (0, 1). Find the probability density function $f_{X_{(k)}}(x)$ of $X_{(k)}$ and compute the expectation $\mathbb{E}X_{(k)}$.
- 10. Consider bond percolation on the square lattice \mathbb{Z}^2 , with every bond independently open with probability $p \in [0, 1]$.
 - (a) Carefully define the percolation probability $\theta(p)$ and show that it is a nondecreasing function of p; hence define the critical value p_{c} .
 - (b) Show that $\theta(p) = 0$ for p > 0 small enough; hence deduce that $p_{c} \ge p'$ for some p' > 0.
 - (c) Show that $\theta(p) > 0$ for 1 p > 0 small enough; hence deduce that $p_{\mathsf{c}} \leq p''$ for some p'' < 1.



SECTION C

- 11. (a) Carefully state Cramér's theorem on large deviations for sums of i.i.d. random variables.
 - (b) Show that under the conditions of Cramér's theorem the rate function I satisfies the inequality $I(a) \ge 0$ for all $a \in \mathbb{R}$; for which value(s) of a does equality hold?

Let X_1, X_2, \ldots be independent Poisson random variables with parameter $\lambda > 0$, and let $S_n = X_1 + \cdots + X_n$.

(c) Compute the moment generating function

$$\varphi(t) := \mathbb{E} \exp\{tX_1\}$$

and the corresponding rate function I(a).

- (d) Verify in this example that $I''(\mathbb{E}X_1) = 1/\mathbb{V}arX_1$.
- (e) Find the limit

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \ge an)$$

for $a > \mathbb{E}X_1$. What happens when $a \leq \mathbb{E}X_1$?