Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH4141-WE01

Title: Geometry IV

Time Allowed:	3 hours	
Additional Material provided:	Formula Sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. (a) Let R_A and R_B be rotations of the Euclidean plane through $\pi/2$ (in positive direction) around the points $A, B \in \mathbb{E}^2$. Describe the composition $f = R_B \circ R_A$.

   (b) Given an orientation-preserving isometry $f \in Isom(\mathbb{E}^2)$, is it always possible to write f as a composition of several rotations through right angles? Justify your answer.

   (c) Let $G \subset Isom(\mathbb{E}^2)$ be the subgroup generated by all rotations through $\pi/2$. Does G act discretely on $\mathbb{E}^2$? Justify your answer.

2. (a) Let f be an affine transformation of the plane $\mathbb{E}^2$ fixing the points $O = (0, 0)$ and $P = (1, 0)$ and mapping the point $(0, 1)$ to $(b, 1)$, where $b \in \mathbb{R}$. Show that f preserves areas of triangles with a vertex at O.

   (b) Let $O = (0, 0)$, $A_1 = (2, 0)$ and $A_2 = (0, 4)$ be points in $\mathbb{E}^2$. Let $A_3, \dots, A_n \in \mathbb{E}^2$ be points such that the line $A_{i-1}A_{i+1}$ is parallel to OA_i, $i = 2, \dots, n-1$. Is it possible that $A_k = (3, 0)$ and $A_{k+1} = (0, 3)$ for some k? Justify your answer.

3. (a) Let $\Delta \subset \mathbb{E}^2$ be a regular triangle inscribed into a circle $\mathcal{C}$. Let I be an inversion with respect to $\mathcal{C}$. Describe the image $I(\Delta)$.

   (b) Let $A_1A_2A_3A_4$ be a quadrilateral on the Euclidean plane. Let $\mathcal{C}_i$ be a circle through the points A_j, A_k, A_l, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Show that the angle between $\mathcal{C}_1$ and $\mathcal{C}_2$ is equal to the angle between $\mathcal{C}_3$ and $\mathcal{C}_4$.

4. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three pairwise non-collinear vectors in $\mathbb{E}^2$ such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = 0,$$

   for $k_1, k_2, k_3 \neq 0$.

   (a) Prove that if these vectors can be affinely transformed into vectors of equal lengths then there exists a triangle with sides $|k_1|, |k_2|, |k_3|$.

   (b) Is the converse statement true, i.e. given that there exists a triangle with sides $|k_1|, |k_2|, |k_3|$, is it always possible to find an affine transformation which maps the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to three vectors of equal lengths? Justify your answer.

5. (a) Let $a, b, c \in (0, 2\pi)$ be positive numbers such that $a < b < c$. Let $ABCD$ and $KLMN$ be quadrilaterals on a unit sphere. Assume that

$$\angle ABC = \angle ACD = \angle KLM = \angle KMN = \pi/2$$

and that

$$AB = NM = a, \quad BC = ML = b, \quad CD = KL = c.$$

What is larger, AD or KN? Justify your answer.

(b) Let $ABCE$ be a spherical quadrilateral with right angles $\angle ABC$ and $\angle BCE$ and with sides $AB = a$, $BC = b$, $CE = c$. Find AE.

6. (a) Let $l \subset \mathbb{E}^2$ be a line and let γ be a circle tangent to l. Let $\mathcal{C}_i$, $i \in \mathbb{Z}$ be circles tangent both to l and γ. Show that there is a circle $\mathcal{C}^{\perp}$ orthogonal to all circles $\mathcal{C}_i$, $i \in \mathbb{Z}$ simultaneously.

(b) In the assumptions of part (a), suppose also that $\mathcal{C}_i$ is tangent to $\mathcal{C}_{i+1}$ for $i \in \mathbb{Z}$. Fix the line and two circles on the complex plane by

$$l = \{\text{Im } z = 0\}, \qquad \gamma = \{|z - \frac{i}{2}| = \frac{1}{2}\}, \qquad \mathcal{C}_0 = \{|z - (1 + \frac{i}{2})| = \frac{1}{2}\}$$

and assume that $\mathcal{C}_1 \subset \{\text{ Im } z < \frac{1}{2}\}$. Find a Möbius transformation f such that $f(\mathcal{C}_i) = \mathcal{C}_{i+1}$ for all $i \in \mathbb{Z}$.

SECTION B

7. A polygon P on the hyperbolic plane is called *ideal* if all vertices of P lie on the absolute.

   (a) Show that for every $\alpha \in (0, \pi)$ there exists an ideal quadrilateral with diagonals intersecting each other at angle α.

   (b) Show that any two ideal quadrilaterals with diagonals intersecting at the same angle are congruent.

   (c) Show that in any ideal quadrilateral the two common perpendiculars to its opposite sides are orthogonal to each other.

   (d) Let P_n be a regular ideal n-gon and $\mathcal{C}_n$ be the circle inscribed in P_n, i.e. $\mathcal{C}_n$ is tangent to all sides of P_n. Denote by S_D the area of a domain D.
   Show that $\frac{S_{\mathcal{C}_n}}{S_{P_n}} \to 0$ as $n \to \infty$.
   You may use that $2\sinh^2 \frac{x}{2} = \cosh x - 1$.

8. (a) The points $A_1, B_1, C_1, D_1 \in \mathbb{R}P^2$ are given in homogeneous coordinates by

   $$A_1 = (1:2:1), \quad B_1 = (2:1:1), \quad C_1 = (3:3:2), \quad D_1 = (1:-1:0).$$

   Is there a projective transformation of $\mathbb{R}P^2$ which maps the points A_1, B_1, C_1, D_1 to

   $$A_2 = (0:0:1), \quad B_2 = (1:0:0), \quad C_2 = (1:0:1), \quad D_2 = (1:1:0)$$

   respectively? Justify your answer.

   (b) Is there a projective transformation of $\mathbb{R}P^2$ which maps the points A_1, B_1, C_1, D_1 defined in part (a) to

   $$A_3 = (0:0:1), \quad B_3 = (0:1:1), \quad C_3 = (0:2:1), \quad D_3 = (0:3:1)$$

   respectively? Justify your answer.

   (c) Let $A, B, C, D \in \mathbb{R}P^2$ be four non-collinear points such that A, B, C are collinear. Is it true that there exists a projective transformation mapping A, B, C, D to any other given quadruple $A', B', C', D' \in \mathbb{R}P^2$ of non-collinear points such that A', B', C' are collinear? Justify your answer.

   (d) Four planes pass through a common line $l \subset \mathbb{R}^3$. Let m_1 and m_2 be two lines intersecting the four planes at points P_1, Q_1, R_1, S_1 and P_2, Q_2, R_2, S_2 respectively. Prove that the cross-ratio $[P_1, Q_1, R_1, S_1]$ is equal to the cross-ratio $[P_2, Q_2, R_2, S_2]$.

9. Let $ABCD \subset \mathbb{E}^2$ be a quadrilateral with AD parallel to BC, $|BC| < |AD|$. Let $E = AB \cap CD$ and $F = AC \cap BD$. Let $Q = EF \cap AD$.

(a) In what proportion does Q divide AD? If in addition $|AE| = 2|AB|$, in what proportion does F divide EQ? Justify your answer.

(b) Let $A_1, \dots, A_{k+1} \in AE$ be points such that $A_{k+1} := A$ and $|EA_i| = i|EA_1|$ for all $i \in \{1, 2, \dots, k+1\}$. Similarly, let $D_1, \dots, D_{k+1} \in DE$ be points such that $D_{k+1} := D$ and $|ED_i| = i|ED_1|$ for all $i \in \{1, 2, \dots, k+1\}$. Are the lines A_iD_{i+1}, for $i = 1, \dots, k$, parallel to each other? Justify your answer.

(c) Let $N = EQ \cap A_kB_{k+1}$. Find the proportion $|NQ|/|EQ|$.

(d) Given the points $X_1, \dots, X_k \in EA$ and $Y_1, \dots, Y_k \in ED$, denote

$$Z_i = X_iY_{i+1} \cap Y_iX_{i+1}, \qquad i = 1, \dots, k-1.$$

Denote also by M the midpoint of X_1Y_1.
Is it true that the segments X_iY_i, $i = 1, \dots, k$, are parallel to each other if and only if the points $E, M, Z_1, \dots, Z_k$ are collinear? Justify your answer.

10. Given two sets X and Y in the hyperbolic plane, denote by $d(X, Y) = \inf\limits_{x \in X, y \in Y} d(x, y)$ the distance between these sets.

(a) Let $l \subset \mathbb{H}^2$ be a line and $A \in \mathbb{H}^2$ be a point such that $d(A, l) = q$. Let e_l be an equidistant curve to l such that $d(e_l, l) = s$. Find the distance $d(A, e_l)$.

(b) Let $l, m \subset \mathbb{H}^2$ be two diverging lines, $d(l, m) = q$. Let e_l and e_m be equidistant curves to the lines l and m respectively, $d(e_l, l) = s$ and $d(e_m, m) = t$.
Find the distance $d(e_l, e_m)$ between the equidistant curves e_l and e_m.

(c) Let l and m be two diverging lines in the hyperbolic plane. Describe the set of points $\{p \in \mathbb{H}^2 \mid d(p, m) = d(p, l)\}$.

(d) Let l_1, l_2, l_3 be three mutually diverging lines in the hyperbolic plane. Suppose that none of the three lines separates the other two. Is it always possible to find a circle tangent to all three lines l_1, l_2, l_3 simultaneously? Justify your answer.

SECTION C

11. (a) Give the definition of an ellipse. Write an affine transformation A which takes the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to a unit circle.

    (b) Let $F_1 = (-f, 0)$ and $F_2 = (f, 0)$ be the foci of an ellipse E. Given a point $P = (p_1, p_2) \in E$ on the ellipse, find the numbers a and b such that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is isometric to E.

    (c) Let E be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let $A = (a, 0)$ and $B = (0, b)$ be points on E. Find a point $X \in E$ such that the tangent to the ellipse E at the point X is parallel to AB. Is such a point unique?

    (d) Let E be an ellipse and F_1 be one of its foci. Let P, Q be points on E. Is there another ellipse $E' \neq E$ such that $P, Q \in E'$ and F_1 is a focus of E'?
    Justify your answer.