Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH4151-WE01

Title:
Elliptic Functions IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. (a) Carefully draw the standard fundamental domain $\mathcal{F}$ inside the upper half plane $\mathbb{H}$ for $\mathrm{SL}_2(\mathbb{Z})$. Specify the identifications at the boundary of $\mathcal{F}$ and how they arise.

   (b) Find an element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ which maps $\tau_0 = (2i + 16)/5$ to $\mathcal{F}$. What is the point in $\mathcal{F}$ equivalent to τ_0?

2. (a) Carefully state the valence formula.

   (b) Recall that the modular j-function is given by $j(\tau) = E_4^3(\tau)/\Delta(\tau)$. Here E_4 is the normalized Eisenstein series of weight 4 and Δ is the discriminant function. Consider $j(\tau) - c$ with $c \in \mathbb{C}$ to show that the j-function defines a bijection from $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ to the complex numbers.

3. (a) Carefully define the Riemann zeta function $\zeta(s)$ and also its completion $Z(s)$. State the fundamental analytic properties of $Z(s)$.

   (b) Compute $\zeta(-1)$. [You may appeal to the ζ-values at the even positive integers, the functional equation $\Gamma(s+1) = s\Gamma(s)$ of the Γ-function and its special values such as $\Gamma(1/2) = \sqrt{\pi}$.]

4. (a) State the definition of an **elliptic function**.

   (b) Let $\{\omega_1, \omega_2\}$ be a basis of a lattice $\Omega \subset \mathbb{C}$. Suppose that $f(z)$ is an even elliptic function with the period lattice Ω and with only one pole at $z = 0 \pmod{\Omega}$ whose order is two. Show that $f'(z)$ is an odd elliptic function which only vanishes at $z = \frac{\omega_1}{2}$, $z = \frac{\omega_2}{2}$ and $z = \frac{\omega_1+\omega_2}{2} \pmod{\Omega}$.

5. (a) State the definition of the **order** of an elliptic function $f(z)$ with the period lattice Ω.

   (b) Suppose that $f(z)$ is an elliptic function of order $N > 0$. For any polynomial $P(T)$ in T of degree $d > 0$, show that $P(f(z))$ is an elliptic function and find its order.

   (c) Suppose that $f(z)$ is an elliptic function of order $N > 0$. Show that $f''(z)$ is an elliptic function of order M with $N + 2 \leq M \leq 3N$.

6. Let $\{\omega_1, \omega_2\}$ be a basis of a lattice $\Omega \subset \mathbb{C}$ with $\mathrm{Im}(\frac{\omega_2}{\omega_1}) > 0$. Let $\zeta(z)$ be the Weierstrass ζ-function associated to Ω.

   (a) Prove the pseudoperiodicity formula for $\zeta(z)$:

   $$\zeta(z + \omega_j) = \zeta(z) + \eta_j \quad (j = 1, 2),$$

   where $\eta_j = 2\zeta(\frac{\omega_j}{2})$.

   (b) Prove Legendre's relation:

   $$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i.$$

SECTION B

7. Recall that the Eisenstein series $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n$ satisfies the transformation property $E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}$. Here as always $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau}$.

   (a) Consider the differential operator $D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$. Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, the space of modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Show that

$$(Df)(\gamma\tau) = (c\tau + d)^{k+2}(Df)(\tau) + \frac{1}{2\pi i}kc(c\tau + d)^{k+1}f(\tau)$$

   for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. [Hint: Consider $\frac{\partial}{\partial\tau}(f(\gamma\tau))$.]

   (b) Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. Show that

$$Df - \tfrac{k}{12}E_2 f \in M_{k+2}(\mathrm{SL}_2(\mathbb{Z})).$$

   Moreover, if f is a cusp form, then so is $Df - \frac{k}{12}E_2 f$.

   (c) Apply (b) to the discriminant function $\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n)q^n$ to show for all n that

$$(1 - n)\tau(n) = 24\sum_{k=1}^{n-1} \sigma_1(k)\tau(n - k).$$

   Carefully state what results on the spaces of modular forms you are using.

8. (a) Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_0(4))$ be a modular form of even weight k and level 4. We define the operator U_2 by

$$U_2(f)(\tau) := \frac{1}{2}\left[f\left(\tau/2\right) + f\left((\tau + 1)/2\right)\right].$$

   Show that $U_2(f)(\tau) \in M_k(\Gamma_0(4))$. Also show that

$$U_2(f)(\tau) = \sum_{n=0}^{\infty} a_{2n}q^n.$$

   [You may assume that $\Gamma_0(4)$ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$, and $\pm I$. At some point $\frac{\tau}{8\tau+2} = \frac{\tau/2}{8(\tau/2)+1}$ and $\frac{5\tau+1}{8\tau+2} = \frac{5(\tau+1)/2-2}{8(\tau+1)/2-3}$ might be helpful.]

   (b) Specify to $k = 2$. You may assume that the theta series $\theta^4(\tau) = \sum_{n=0}^{\infty} r_4(n)q^n$ and $F(\tau) = \sum_{\substack{m>0 \\ m\,\mathrm{odd}}} \sigma_1(m)q^m$ form a basis of $M_2(\Gamma_0(4))$.
   Compute $U_2(\theta^4)$ and $U_2(F)$ and conclude that F and $\theta^4 + 16F$ form an eigenbasis for U_2 on $M_2(\Gamma_0(4))$. What are the eigenvalues? Note that they are different.

   (c) You may assume that the Hecke operator T_p on $M_2(\Gamma_0(4))$ for an *odd* prime p is given by $T_p(f)(\tau) = \sum_{n=0}^{\infty}(a_{pn} + pa_{n/p})q^n$. Note that U_2 commutes with all those T_p.
   Use (b) to show that F and $\theta^4 + 16F$ are also eigenfunctions for all those T_p. Compute the eigenvalues (separately) and note that they are the same!

   (d) Conclude that θ^4 is also an eigenform for all T_p for p odd and obtain $r_4(p) = 8(p + 1)$ for all odd primes p.

9. Let $\Omega \subset \mathbb{C}$ be a lattice with a basis $\{\omega_1, \omega_2\}$. Let $\sigma(z)$ be the Weierstrass σ-function associated to Ω.

   (a) Use the pseudoperiodicity formula for $\zeta(z)$ (see **Question 6(a)**) to deduce the pseudoperiodicity formula for $\sigma(z)$:

$$\sigma(z+\omega_j) = -\exp\left(\eta_j\left(z+\frac{\omega_j}{2}\right)\right)\sigma(z) \quad (j=1,2).$$

   (b) Fix a number $u \in \mathbb{C}$ such that $2u \notin \Omega$. Let

$$f(z) = \frac{\sigma(z-u)\sigma(z+u)}{(\sigma(z)\sigma(u))^2}.$$

   Prove that $f(z)$ is an elliptic function whose period lattice agrees with Ω, with simple zeros at $z = \pm u \pmod{\Omega}$ and a pole of order two at $z = 0 \pmod{\Omega}$.

   (c) Let $f(z)$ be as in (b). Show that there exists a constant A such that

$$f(z) = A\left(\wp(z) - \wp(u)\right),$$

   where $\wp(z)$ is the Weierstrass $\wp$-function associated to Ω.

   (d) Find the value of A in (c).

10. Let $\Omega \subset \mathbb{C}$ be a lattice. Let $\zeta(z)$ be the Weierstrass ζ-function associated to Ω. Fix a number $u \in \mathbb{C}$ such that $2u \notin \Omega$.

   (a) Use the pseudoperiodicity formula for $\zeta(z)$ (see **Question 6(a)**) to show that the function

$$\zeta(z+u) + \zeta(z-u) - 2\zeta(z)$$

   is an odd elliptic function with period lattice Ω and simple poles at $z = 0, \pm u$.

   (b) By comparing the residue at each pole or other methods, show that the function

$$f(z) := \zeta(z+u) + \zeta(z-u) - 2\zeta(z) - \frac{\wp'(z)}{\wp(z) - \wp(u)}$$

   is a constant function. Here $\wp(z)$ is the Weierstrass $\wp$-function associated to Ω.

   (c) By considering the zeros of $\zeta(z+u) + \zeta(z-u) - 2\zeta(z)$ or by other methods, show that the function $f(z)$ in (b) is identically zero, or equivalently,

$$\zeta(z+u) + \zeta(z-u) - 2\zeta(z) = \frac{\wp'(z)}{\wp(z) - \wp(u)}.$$

   You may use the statement of **Question 4(b)** as a hint for finding zeros.

   (d) Prove the following formula:

$$\zeta(z+u) = \zeta(z) + \zeta(u) + \frac{1}{2}\frac{\zeta''(z) - \zeta''(u)}{\zeta'(z) - \zeta'(u)}.$$