

## EXAMINATION PAPER

Examination Session: May

2017

Year:

Exam Code:

MATH4151-WE01

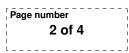
Title:

Elliptic Functions IV

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for: the best <b>FOUR</b> answers from Section and the best <b>THREE</b> answers from S Questions in Section B carry <b>TWICE</b> in Section A.	ection B. as many ma	arks as those
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Revision:



## SECTION A

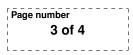
- 1. (a) Carefully draw the standard fundamental domain  $\mathcal{F}$  inside the upper half plane  $\mathbb{H}$  for  $\mathrm{SL}_2(\mathbb{Z})$ . Specify the identifications at the boundary of  $\mathcal{F}$  and how they arise.
  - (b) Find an element  $\gamma \in \text{SL}_2(\mathbb{Z})$  which maps  $\tau_0 = (2i + 16)/5$  to  $\mathcal{F}$ . What is the point in  $\mathcal{F}$  equivalent to  $\tau_0$ ?
- 2. (a) Carefully state the valence formula.
  - (b) Recall that the modular *j*-function is given by  $j(\tau) = E_4^3(\tau)/\Delta(\tau)$ . Here  $E_4$  is the normalized Eisenstein series of weight 4 and  $\Delta$  is the discriminant function. Consider  $j(\tau) - c$  with  $c \in \mathbb{C}$  to show that the *j*-function defines a bijection from  $SL_2(\mathbb{Z})\backslash\mathbb{H}$  to the complex numbers.
- 3. (a) Carefully define the Riemann zeta function  $\zeta(s)$  and also its completion Z(s). State the fundamental analytic properties of Z(s).
  - (b) Compute  $\zeta(-1)$ . [You may appeal to the  $\zeta$ -values at the even positive integers, the functional equation  $\Gamma(s+1) = s\Gamma(s)$  of the  $\Gamma$ -function and its special values such as  $\Gamma(1/2) = \sqrt{\pi}$ .]
- 4. (a) State the definition of an elliptic function.
  - (b) Let  $\{\omega_1, \omega_2\}$  be a basis of a lattice  $\Omega \subset \mathbb{C}$ . Suppose that f(z) is an even elliptic function with the period lattice  $\Omega$  and with only one pole at  $z = 0 \pmod{\Omega}$  whose order is two. Show that f'(z) is an odd elliptic function which only vanishes at  $z = \frac{\omega_1}{2}$ ,  $z = \frac{\omega_2}{2}$  and  $z = \frac{\omega_1 + \omega_2}{2} \pmod{\Omega}$ .
- 5. (a) State the definition of the **order** of an elliptic function f(z) with the period lattice  $\Omega$ .
  - (b) Suppose that f(z) is an elliptic function of order N > 0. For any polynomial P(T) in T of degree d > 0, show that P(f(z)) is an elliptic function and find its order.
  - (c) Suppose that f(z) is an elliptic function of order N > 0. Show that f''(z) is an elliptic function of order M with  $N + 2 \le M \le 3N$ .
- 6. Let  $\{\omega_1, \omega_2\}$  be a basis of a lattice  $\Omega \subset \mathbb{C}$  with  $\operatorname{Im}(\frac{\omega_2}{\omega_1}) > 0$ . Let  $\zeta(z)$  be the Weierstrass  $\zeta$ -function associated to  $\Omega$ .
  - (a) Prove the pseudoperiodicity formula for  $\zeta(z)$ :

$$\zeta(z+\omega_j) = \zeta(z) + \eta_j \quad (j=1,2),$$

where  $\eta_j = 2\zeta(\frac{\omega_j}{2})$ .

(b) Prove Legendre's relation:

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i.$$





## SECTION B

- 7. Recall that the Eisenstein series  $E_2(\tau) = 1 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$  satisfies the transformation property  $E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}$ . Here as always  $\tau \in \mathbb{H}$  and  $q = e^{2\pi i \tau}$ .
  - (a) Consider the differential operator  $D = \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}$ . Let  $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ , the space of modular forms of weight k for  $\mathrm{SL}_2(\mathbb{Z})$ . Show that

$$(Df)(\gamma\tau) = (c\tau + d)^{k+2}(Df)(\tau) + \frac{1}{2\pi i}kc(c\tau + d)^{k+1}f(\tau)$$

for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ . [Hint: Consider  $\frac{\partial}{\partial \tau} (f(\gamma \tau))$ .]

(b) Let  $f \in M_k(SL_2(\mathbb{Z}))$ . Show that

$$Df - \frac{k}{12}E_2f \in M_{k+2}(\mathrm{SL}_2(\mathbb{Z})).$$

Moreover, if f is a cusp form, then so is  $Df - \frac{k}{12}E_2f$ .

(c) Apply (b) to the discriminant function  $\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n)q^n$  to show for all n that

$$(1-n)\tau(n) = 24\sum_{k=1}^{n-1}\sigma_1(k)\tau(n-k).$$

Carefully state what results on the spaces of modular forms you are using.

8. (a) Let  $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_0(4))$  be a modular form of even weight k and level 4. We define the operator  $U_2$  by

$$U_2(f)(\tau) := \frac{1}{2} \left[ f(\tau/2) + f((\tau+1)/2) \right].$$

Show that  $U_2(f)(\tau) \in M_k(\Gamma_0(4))$ . Also show that

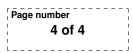
$$U_2(f)(\tau) = \sum_{n=0}^{\infty} a_{2n} q^n.$$

[You may assume that  $\Gamma_0(4)$  is generated by  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ , and  $\pm I$ . At some point  $\frac{\tau}{8\tau+2} = \frac{\tau/2}{8(\tau/2)+1}$  and  $\frac{5\tau+1}{8\tau+2} = \frac{5(\tau+1)/2-2}{8(\tau+1)/2-3}$  might be helpful.]

- (b) Specify to k = 2. You may assume that the theta series  $\theta^4(\tau) = \sum_{n=0}^{\infty} r_4(n)q^n$ and  $F(\tau) = \sum_{\substack{m>0 \ modd}} \sigma_1(m)q^m$  form a basis of  $M_2(\Gamma_0(4))$ . Compute  $U_2(\theta^4)$  and  $U_2(F)$  and conclude that F and  $\theta^4 + 16F$  form an eigenbasis for  $U_2$  on  $M_2(\Gamma_0(4))$ . What are the eigenvalues? Note that they are different.
- (c) You may assume that the Hecke operator  $T_p$  on  $M_2(\Gamma_0(4))$  for an *odd* prime p is given by  $T_p(f)(\tau) = \sum_{n=0}^{\infty} (a_{pn} + pa_{n/p})q^n$ . Note that  $U_2$  commutes with all those  $T_p$ .

Use (b) to show that F and  $\theta^4 + 16F$  are also eigenfunctions for all those  $T_p$ . Compute the eigenvalues (separately) and note that they are the same!

(d) Conclude that  $\theta^4$  is also an eigenform for all  $T_p$  for p odd and obtain  $r_4(p) = 8(p+1)$  for all odd primes p.





- 9. Let  $\Omega \subset \mathbb{C}$  be a lattice with a basis  $\{\omega_1, \omega_2\}$ . Let  $\sigma(z)$  be the Weierstrass  $\sigma$ -function associated to  $\Omega$ .
  - (a) Use the pseudoperiodicity formula for  $\zeta(z)$  (see **Question 6(a)**) to deduce the pseudoperiodicity formula for  $\sigma(z)$ :

$$\sigma(z+\omega_j) = -\exp\left(\eta_j\left(z+\frac{\omega_j}{2}\right)\right)\sigma(z) \quad (j=1,2).$$

(b) Fix a number  $u \in \mathbb{C}$  such that  $2u \notin \Omega$ . Let

$$f(z) = \frac{\sigma(z-u)\sigma(z+u)}{(\sigma(z)\sigma(u))^2}.$$

Prove that f(z) is an elliptic function whose period lattice agrees with  $\Omega$ , with simple zeros at  $z = \pm u \pmod{\Omega}$  and a pole of order two at  $z = 0 \pmod{\Omega}$ .

(c) Let f(z) be as in (b). Show that there exists a constant A such that

$$f(z) = A\left(\wp(z) - \wp(u)\right),$$

where  $\wp(z)$  is the Weierstrass  $\wp$ -function associated to  $\Omega$ .

- (d) Find the value of A in (c).
- 10. Let  $\Omega \subset \mathbb{C}$  be a lattice. Let  $\zeta(z)$  be the Weierstrass  $\zeta$ -function associated to  $\Omega$ . Fix a number  $u \in \mathbb{C}$  such that  $2u \notin \Omega$ .
  - (a) Use the pseudoperiodicity formula for  $\zeta(z)$  (see **Question 6(a)**) to show that the function

$$\zeta(z+u) + \zeta(z-u) - 2\zeta(z)$$

is an odd elliptic function with period lattice  $\Omega$  and simple poles at  $z = 0, \pm u$ .

(b) By comparing the residue at each pole or other methods, show that the function

$$f(z) := \zeta(z+u) + \zeta(z-u) - 2\zeta(z) - \frac{\wp'(z)}{\wp(z) - \wp(u)}$$

is a constant function. Here  $\wp(z)$  is the Weierstrass  $\wp$ -function associated to  $\Omega$ .

(c) By considering the zeros of  $\zeta(z+u) + \zeta(z-u) - 2\zeta(z)$  or by other methods, show that the function f(z) in (b) is identically zero, or equivalently,

$$\zeta(z+u) + \zeta(z-u) - 2\zeta(z) = \frac{\wp'(z)}{\wp(z) - \wp(u)}$$

You may use the statement of Question 4(b) as a hint for finding zeros.

(d) Prove the following formula:

$$\zeta(z+u) = \zeta(z) + \zeta(u) + \frac{1}{2} \frac{\zeta''(z) - \zeta''(u)}{\zeta'(z) - \zeta'(u)}.$$