Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH4161-WE01

Title:

Algebraic Topology IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

The following notations hold in this paper

- $\mathbb{R}^n$ denotes real n-space with the usual topology.
- D^{n+1} and S^n denote the closed unit ball and unit sphere in $\mathbb{R}^{n+1}$ with the subspace topology.
- For n a positive integer, $\mathbb{Z}/n$ denotes the quotient group $\mathbb{Z}/n\mathbb{Z}$.
  Elements of $\mathbb{Z}/n$ are denoted as $\bar{k}$ for $k \in \mathbb{Z}$.

SECTION A

1. (a) Define what is meant by a short exact sequence of abelian groups.

   (b) Give an example of a short exact sequence

   $$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

   where B is not isomorphic to $A \oplus C$.

   (c) Suppose that

   $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

   is a short exact sequence and that there exists an $s : B \to A$ such that $s \circ f$ is the identity map on A. Show that $B \cong A \oplus C$.

2. (a) Define what is meant by a cell complex.

   (b) Suppose that X is a topological space admitting a cell decomposition whose only non-zero homology groups are

   $$H_0(X) = \mathbb{Z} \oplus \mathbb{Z}, H_1(X) = (\mathbb{Z}/2), H_2(X) = \mathbb{Z}, H_3(X) = \mathbb{Z} \oplus (\mathbb{Z}/3).$$

   Show that any cell decomposition of X has at least 8 cells.

3. For any point $p \in \mathbb{R}^3$ we write $S^2(p)$ for the 2-sphere formed of all points which are distance 1 away from the point p.

   Using any method that you like, determine the homology groups of the space $X \subset \mathbb{R}^3$ where

   $$\begin{aligned} X \;=\; & S^2(1,1,1) \cup S^2(1,1,-1) \cup S^2(1,-1,1) \cup S^2(1,-1,-1) \cup \\ & S^2(-1,1,1) \cup S^2(-1,1,-1) \cup S^2(-1,-1,1) \cup S^2(-1,-1,-1). \end{aligned}$$

4. For $n \geq 0$ let $C_n = \mathbb{Z}/32$, and for $n < 0$ let $C_n = 0$. For $n \geq 1$ define $\partial_n \colon C_n \to C_{n-1}$ as multiplication by 8.

   (a) Show that (C_*, ∂_*) is a chain complex and calculate its homology.

   (b) Calculate $H^k(\mathrm{Hom}(C_*, \mathbb{Z}/16))$ and $H^k(\mathrm{Hom}(C_*, \mathbb{Z}/64))$ for all $k \geq 0$.

5. (a) State the Universal Coefficient Theorem.

   (b) Using the naturality of the Universal Coefficient Theorem or otherwise, show that every map $f \colon S^2 \to \mathbb{R}\mathbf{P}^2$ induces the zero homomorphism

$$f^* \colon H^2(\mathbb{R}\mathbf{P}^2; \mathbb{Z}/2) \to H^2(S^2; \mathbb{Z}/2).$$

6. (i) Let X be a topological space, $k, l \geq 0$ and $\varphi \in C^k(X;\mathbb{Z})$ and $\psi \in C^l(X;\mathbb{Z})$ singular cochains. State the formula for the cup-product $\varphi \cup \psi \in C^{k+l}(X;\mathbb{Z})$.

   (ii) Let Y be a finite cell complex whose homology is given by

$$H_k(Y) = \begin{cases} \mathbb{Z} & k = 0, 5 \\ \mathbb{Z}/4 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Calculate $H^k(Y; \mathbb{Z}/4)$ for $k \geq 0$.

   (b) List the possible cohomology ring structures $H^*(Y; \mathbb{Z}/4)$.

SECTION B

7. (a) The diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & A_{-1} & \longrightarrow & \cdots \\
 & & {\scriptstyle \phi_2}\big\downarrow{\scriptstyle \psi_2} & & {\scriptstyle \phi_1}\big\downarrow{\scriptstyle \psi_1} & & {\scriptstyle \phi_0}\big\downarrow{\scriptstyle \psi_0} & & {\scriptstyle \phi_{-1}}\big\downarrow{\scriptstyle \psi_{-1}} & & \\
\cdots & \longrightarrow & B_2 & \longrightarrow & B_1 & \longrightarrow & B_0 & \longrightarrow & B_{-1} & \longrightarrow & \cdots
\end{array}$$

shows two chain maps ϕ_* and ψ_* between the chain complexes A_* and B_*. Define what is meant by a *chain homotopy* between ϕ_* and ψ_*, and show that the existence of a chain homotopy implies that ϕ_* and ψ_* induce the same maps on homology.

(b) Suppose that

$$\begin{array}{ccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{f_1} & A_0 & \longrightarrow & 0 \\
 & & {\scriptstyle \phi_1}\big\downarrow & & \big\downarrow{\scriptstyle \phi_0} & & \\
0 & \longrightarrow & B_1 & \xrightarrow[g_1]{} & B_0 & \longrightarrow & 0
\end{array}$$

is a commutative diagram. We think of

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_0 \longrightarrow 0$$

and

$$0 \longrightarrow B_1 \xrightarrow[g_1]{} B_0 \longrightarrow 0$$

as chain complexes and of (ϕ_1, ϕ_0) as being a chain map. Show that if the map induced on homology in degree 0 by this chain map is the zero map and A_0 is finitely generated and free abelian, then there exists a map $h : A_0 \to B_1$ such that $g_1 \circ h = \phi_0$.

(c) It is a reasonable conjecture that if the maps induced on homology in degrees 0 and 1 by (ϕ_1, ϕ_0) are *both* the zero map then there exists a map $h : A_0 \to B_1$ such that both $g_1 \circ h = \phi_0$ and $h \circ f_1 = \phi_1$. Show that this conjecture is false by giving a counterexample in which each of A_0, A_1, B_0, B_1 are finitely generated and free abelian.

8. Given a topological space X and a map $f : X \to X$, we form the *mapping torus* T_f from $X \times [0,1]$ by gluing $X \times \{0\}$ to $X \times \{1\}$ using f as the gluing map:

$$T_f = (X \times [0,1])/(x,0) \sim (f(x),1).$$

(a) State the Mayer-Vietoris theorem.

(b) Let X be such that the homology groups of X agree with the homology groups of a point, and let $f : X \to X$ be any homeomorphism of X. Using the Mayer-Vietoris theorem or otherwise, show that the homology groups of T_f agree with the homology groups of S^1.

(c) Now suppose that $X = S^1 \subset \mathbb{C}$ is the unit circle

$$X = \{z \in \mathbb{C} : |z| = 1\}$$

and $f : z \mapsto z^5$. Compute the homology groups of T_f.

9. Let X be the quotient space obtained from $[0,1] \times [0,1]$ using the identifications on the boundary indicated in Figure 1, that is, we identify the points $(x,0) \sim (1,x) \sim (1-x,1) \sim (0,1-x)$ for $x \in [0,1]$.

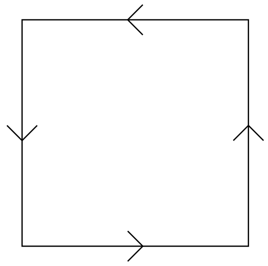

Figure 1: X as a square with identifications on the boundary.

(a) Calculate $H_k(X)$ for all $k \geq 0$.

(b) Calculate $H^k(X;\mathbb{Z}/2)$ and $H^k(X;\mathbb{Z}/4)$ for all $k \geq 0$.

(c) Draw a triangulation of X. Hint: You may want to use four vertices for the circle represented by $(x,0)$, $x \in [0,1]$ rather than only three.

(d) Denote by $C_*^{\Delta}(X)$ the simplicial chain complex obtained from your triangulation. For every non-zero element $\alpha \in H^1(X;\mathbb{Z}/2)$ write down a cocycle $A \in \mathrm{Hom}(C_1^{\Delta}(X), \mathbb{Z}/2)$ which represents α. Hint: Use the subspace Y of X indicated by the dotted lines in Figure 2.

(e) Determine the cohomology ring $H^*(X;\mathbb{Z}/2)$.

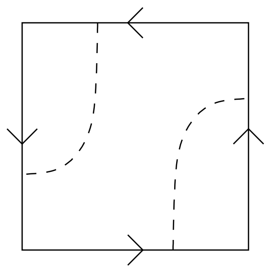

Figure 2: X together with subspace Y.

10. (i) Let M be a closed 4-dimensional manifold which admits a triangulation and with $H_0(M) \cong \mathbb{Z} \cong H_4(M)$.

(a) State the Poincaré Duality Theorem for M.

(b) Assume that $H_1(M) = 0$. Show that $H_3(M) = 0$ and $H_2(M)$ does not contain torsion elements.

(c) Let $b\colon H^2(M;\mathbb{Q}) \times H^2(M;\mathbb{Q}) \to \mathbb{Q}$ be defined by

$$b(x,y) = \langle [M], x \cup y \rangle.$$

Show that b is bilinear, non-degenerate and symmetric. Recall that non-degenerate means that for every $x \neq 0$ there exists a y with $b(x,y) \neq 0$. Determine b for $M = S^2 \times S^2$.

(ii) Assume $f\colon \mathbb{R}\mathbf{P}^3 \to S^5$ is an embedding.

(a) State the Alexander Duality Theorem.

(b) Calculate $H_*(S^5 - f(\mathbb{R}\mathbf{P}^3))$.