Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH4171-WE01

Title:

Riemannian Geometry IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. (a) Define what it means that a set M is a manifold of dimension n.

   (b) State the Implicit Function Theorem.

   (c) Let $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + 3x_2^2 + x_3^2 = 1\}$ be an ellipsoid. Prove that M is a 2-dimensional manifold.

2. (a) Let M be a manifold. Define the notion of a vector field X on M.

   (b) Let X and Y be two vector fields on a manifold M. Define the Lie bracket $[X, Y]$ and prove that $[fX, Y] = -(Yf)X + f[X, Y]$ for any smooth function f on M.

3. (a) Define the Levi-Civita connection on a Riemannian manifold (M, g).

   (b) Define what it means that a curve $c : [a, b] \to M$ is a geodesic in (M, g) and prove that a geodesic has constant speed $|c'(t)|$, $t \in [a, b]$.

4. (a) Let (M, g) be a Riemannian manifold and let $M \supset U \ni p \to x \in \mathbb{R}^n$ be local coordinates. Define the Christoffel symbols of (M, g) in terms of the local coordinates x.

   (b) Let $p = (0, 0, 1)$. Compute
$$\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2}(p)$$
   on the manifold
$$M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + 3x_2^2 + x_3^2 = 1\}$$
   with induced metric $g =<, >_{\mathbb{R}^3}$ and local coordinates
   $M \cap \{x_3 > 0\} \to \mathbb{R}^2, (x_1, x_2, x_3) \to (x_1, x_2)$.

5. (a) Let (M, g) be a Riemannian manifold. Define the distance function
   $d : M \times M \to \mathbb{R}$.

   (b) Prove that for all $p, q \in M$ we have $d(p, q) \geq 0, d(p, p) = 0$.

6. (a) Let $M := S^2 = \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$ be the round sphere and $g :=<, >_{\mathbb{R}^3}$ be the metric induced by the Euclidean metric of $\mathbb{R}^3$. Prove that any two points of M are connected by a geodesic.

   (b) State the Theorem of Hopf-Rinow.

SECTION B

7. (a) Define the Riemannian curvature tensor $R(X,Y)Z$ of a Riemannian manifold (M,g). Here, X, Y, Z are three vector fields.

   (b) Let f be a smooth function on M and $R(X,Y)Z$ be as above. Prove that $R(X,Y)fZ = fR(X,Y)Z$.

   (c) Define the Ricci curvature $Ric(X,Y)$ of (M,g) and prove that $Ric(fX,Y) = fRic(X,Y)$ for f as in (b).

   (d) State the Theorem of Bonnet - Myers.

8. (a) Consider $M = \{x \in \mathbb{R}^n | \; |x| < 1\}$ with metric $g_{ij} = \frac{4}{(1-|x|^2)^2}\delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

   . Compute the Christoffel symbols of (M,g)).

   (b) Show that the curve $c(t) = t(1,0,...,0), t \in [0,1)$ in M may be reparametrized to be a geodesic. You may use that the differential equation $u'(s) = \frac{1-u^2(s)}{2}$, $u(0) = 0$ has solution $u(s) = \frac{e^s-1}{e^s+1}$.

   (c) Determine the length of the curve c.

9. (a) Let M be a Riemannian manifold, $p \in M$ and σ be a two-dimensional subspace of T_pM. Define the sectional curvature $K(\sigma)$.

   (b) Define what it means for two points $p, q \in M$ to be conjugate points.

   (c) State the Theorem of Cartan-Hadamard.

   (d) Let $M = \{x \in \mathbb{R}^2 | x_1 > 0\}$ and $g_{11} = 1, g_{12} = g_{21} = 0, g_{22} = (1 + x_1)$. Prove or disprove that (M,g) has non-positive sectional curvature.

10. (a) Consider two Riemannian manifolds $(M_1, g_1), (M_2, g_2)$. Prove that $(M_1 \times M_2, h)$ is a Riemannian manifold, where $T(M_1 \times M_2) = TM_1 \oplus TM_2$ and $h((X_1, X_2), (Y_1, Y_2)) := g_1(X_1, Y_1) + g_2(X_2, Y_2)$.

    (b) Show that on $(M_1 \times M_2, h)$ we have

$$h(R((X_1, X_2), (Y_1, Y_2))(Z_1, Z_2), (W_1, W_2)) = \\ g_1(R_1(X_1, Y_1)Z_1, W_1) + g_2(R_2(X_2, Y_2)Z_2, W_2),$$

    where R_1, R_2 are the curvature tensors of $(M_1, g_1), (M_2, g_2)$, respectively.

    (c) Prove that the Ricci curvature of $(M_1 \times M_2, h)$ satisfies

$$Ric((X_1, X_2), (Y_1, Y_2)) = Ric_1(X_1, Y_1) + Ric_2(X_2, Y_2).$$

    Here, Ric_1, Ric_2 are the Ricci curvatures of $(M_1, g_1), (M_2, g_2)$, respectively.