Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH4181-WE01

Title:
Mathematical Finance IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. A butterfly spread is formed from three types of European call option with the same expiry date T, and different strike prices $K_1 < K_2 < K_3$, by buying one call option with strike price K_1 and one call option with strike price K_3 and selling two call options with strike price $K_2 = \frac{1}{2}(K_1 + K_3)$

   (a) Draw the graph of the payoff at the expiry time of a butterfly spread built from these call options.

   (b) Suppose that the market prices of European call options with expiry time 6 months are as follows:

Strike price	Current price of option
50	20
60	14
70	12

   What is the maximum payoff a butterfly spread created from these options can return?

   (c) Assuming an interest rate of 5% compounded continuously, for what value(s) of the share price in 6 months' time does the butterfly spread in part (b) make a profit?

2. Consider a financial market $\mathcal{M} = (B_t, S_t)$ on two assets. Suppose the current share price of the risky asset is $S_0 = 100$ and evolves according to the binomial model with $u = 1.2$, $d = 0.8$, $p_u = p_d = 1/2$ and suppose that the current price of the risk-free asset is $B_0 = 1$ and the interest rate is $r = 0.1$.

   (a) Calculate the risk-neutral measure for this market.

   (b) Calculate the no-arbitrage prices at times $t = 0, 1, 2$ of a lookback put option on this market with payoff $S_{\max} - S_T$ at time $T = 3$.

3. (a) By considering appropriate portfolios prove that the no-arbitrage price P of a European put option with strike price K and expiry date T satisfies the inequalities
$$(K\mathrm{e}^{-rT} - S)^+ \leq P \leq K\mathrm{e}^{-rT},$$
   where S is the current price of the underlying stock and r is the interest rate, compounded continuously.

   (b) State and prove the corresponding inequalities for the price C of a European call option with the same strike price and expiry date.

4. A stock price follows a geometric Brownian motion with drift $\mu = 0.05$ and volatility $\sigma = 0.3$ and its current price is 95. The risk free interest rate is $r = 0.04$.

   (a) Find the no-arbitrage price at time 0 of a European call option on this stock with strike price $K = 100$ and expiry date $T = 0.25$.

   (b) What is the probability that the call option is worth nothing at the expiry date?

   [The standard normal cdf is tabulated at the bottom of the page.]

5. (i) State the definition of a Brownian motion.

   (ii) Let $(W_t)_{t\geq 0}$ be a Brownian motion. Find the probability density function of the Itô integral $I = \int_0^\pi \cos t \, \mathrm{d}W_t$.

   (iii) Calculate $\mathbb{E}[\int_0^2 W_t^2 \, \mathrm{d}W_t]$ and $\mathbb{V}\mathrm{ar}[\int_0^2 W_t^2 \, \mathrm{d}W_t]$.

6. Suppose that X is a Normal random variable with mean μ and variance σ^2 under the probability measure $\mathbb{P}$, and let $\mathbb{Q}$ be an equivalent measure under which X/σ is a standard Normal random variable.

   (a) Find an expression for the Radon–Nikodym derivative $\dfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ as a function of X.

   (b) Calculate the expectation of $\dfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ under the measure $\mathbb{P}$.

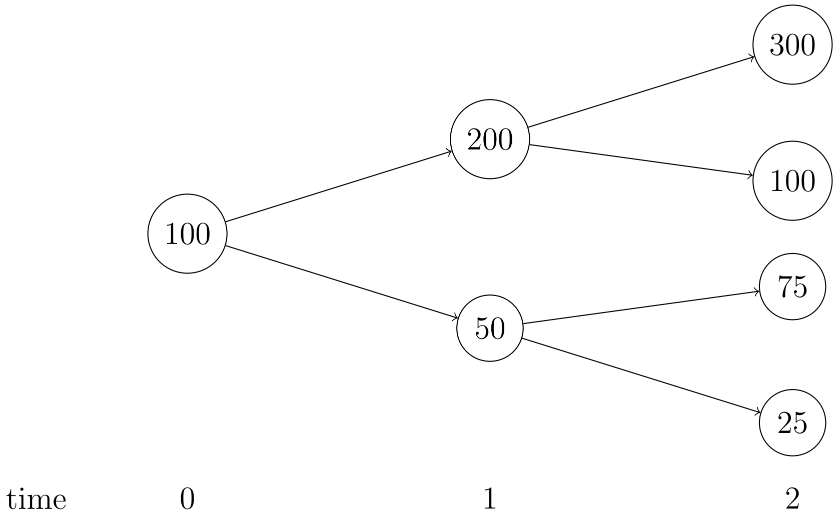
Standard normal cumulative probabilities

z	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$N(z)$	0.540	0.579	0.618	0.655	0.691	0.726	0.758	0.788	0.816	0.841
z	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$N(z)$	0.864	0.885	0.903	0.919	0.933	0.945	0.955	0.964	0.971	0.977
z	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
$N(z)$	0.982	0.986	0.989	0.992	0.994	0.995	0.997	0.997	0.998	0.999

Linear interpolation is accurate to 3 decimal places.

SECTION B

7. Consider a 2-period financial market with possible share prices $S_t, t = 0, 1, 2$, of a risky asset given by the tree:



Suppose the interest rate per time step is $r = 0.1$.

(a) Calculate as a function of K the no-arbitrage price at time 0 of a European put option on this asset with strike price K and expiry time $T = 2$.

(b) Suppose $75 < K < 200$. Calculate as a function of K the no-arbitrage price at time 0 of an American put option on this asset with strike price K and expiry time $T = 2$.

(c) Are there values of the strike price $K > 0$ for which the European and American put options have the same non-zero price?

8. (a) State the definitions of an up-and-in call option and an up-and-out call option on an underlying asset with strike price K, barrier L and expiry date T.

(b) Suppose the current share price of the asset is $S_0 = 80$ and evolves according to the binomial model with $u = 2$, $d = 1/2$, $p_u = p_d = 1/2$ and that the interest rate is $r = 1/4$. Find the no-arbitrage prices at time 0 of both an up-and-in call option and an up-and-out call option on this asset with the same strike price $K = 30$, barrier $L = 200$ and expiry date $T = 3$.

(c) A new "chooser option" is offered on the market based on the above barrier options. The chooser option is sold at time 0 and at time 1 the holder must decide whether the option will be an up-and-in call option, or an up-and-out call option. What is the price at time 0 of this chooser option?

(d) Explain why the price in part (c) is at most the price of a standard call option with the same strike price K and expiry date $T = 3$.

9. (a) State Itô's Lemma for a smooth function $f(t, X_t)$ of time t and an Itô process $(X_t)_{t\geq 0}$.

   (b) Suppose the process $(X_t)_{t\geq 0}$ satisfies a linear stochastic differential equation

$$\mathrm{d}X_t = (a + bX_t)\,\mathrm{d}t + c\,\mathrm{d}W_t, \quad X_0 = 0,$$

   where a, b and c are constants and $(W_t)_{t\geq 0}$ is Brownian motion. Use Itô's Lemma to find a non-zero function $g(t)$ for which $\mathrm{d}(g(t)X_t)$ does not depend on X_t.

   (c) Calculate $\mathbb{E}X_t$ and $\mathbb{V}\mathrm{ar}X_t$.

10. Consider the continuous-time Black–Scholes model, with price dynamics given by

$$\begin{aligned}\mathrm{d}B_t &= rB_t\mathrm{d}t,\\ \mathrm{d}S_t &= \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t,\end{aligned}$$

where $r > 0$ is the risk-free interest rate, μ and σ are constant parameters and $(W_t)_{t\geq 0}$ is a Brownian motion under the real-world measure $\mathbb{P}$.

   (a) Suppose $\widetilde{W}_t = W_t + \theta t$ for some constant θ. Show that there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which $\widetilde{W}_t$ is a Brownian motion, and write down the corresponding Radon–Nikodym derivative.

   (b) State the value of θ that defines the risk-neutral measure, and write down the price dynamics of S_t under the risk-neutral measure.

   (c) Using the risk-neutral valuation formula find an expression for the price at time $t < T$ of the contingent claim with payoff $\Phi(S_T) = (S_T)^\alpha$, for constant α.

   (d) Using your answer to part (c) or otherwise, solve the partial differential equation

$$\frac{\partial F}{\partial t}(t,x) + x\frac{\partial F}{\partial x}(t,x) + 2x^2\frac{\partial^2 F}{\partial x^2}(t,x) = F(t,x)$$

   with terminal condition $F(1, x) = x^\alpha$, for $0 \leq t \leq 1$ and $x \geq 0$.

SECTION C

11. (a) Suppose X is a random variable with unknown mean and variance. Give expressions for the usual unbiased estimators for $\mathbb{E}X$ and $\mathbb{V}\mathrm{ar}X$ based on a sample of size M, and for an approximate $100(1-2\alpha)\%$ confidence interval for $\mathbb{E}X$ using these estimators and the critical value $z_\alpha > 0$ defined by $\mathbb{P}[Z > z_\alpha] = \alpha$, where $Z \sim \mathcal{N}(0,1)$.

(b) What is the approximate probability that the confidence interval contains $\mathbb{E}X$? What assumptions make this approximation valid?

(c) Describe carefully a Monte Carlo algorithm for producing an approximate 95% confidence interval for the fair price at time 0 of a European call option with strike price K and expiry time T on an underlying risky asset with volatility σ, current share price S_0 and interest rate r.

(d) For the parameters $S_0 = 10$, $\sigma = 0.2$, $r = 0.01$, $K = 9$, $T = 1$, use your algorithm to calculate an approximate 95% confidence interval based on the following sample of size 5 from a standard Normal distribution:

$$1.088,\ 0.744,\ -1.724,\ -0.651,\ 2.253$$

[You may find some of the following values of the standard normal cdf useful.]

z	1.28	1.645	1.96
$N(z)$	0.9	0.95	0.975

(e) What change to the confidence interval would you expect if you used a sample of size 50?