Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2017	MATH4221-WE01

Title:
Numerical Differential Equations IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2017

SECTION A

1. (a) Write down the conditions (in terms of the coefficients a_{ij}, b_j and c_j) for a Runge–Kutta scheme to be of order (at least) 3.

   (b) Find all explicit 2-stage Runge–Kutta schemes of order 2.

2. (a) Describe briefly how one obtains the implicit Adams scheme

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + k \sum_{m=0}^{s} B_m \boldsymbol{f}(\boldsymbol{x}^{n-m+1})$$

   for some constant coefficients B_m.

   (b) Verify that the case $s = 2$ gives the scheme

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \frac{k}{12}\,[\,5\boldsymbol{f}(\boldsymbol{x}^{n+1}) + 8\boldsymbol{f}(\boldsymbol{x}^n) - \boldsymbol{f}(\boldsymbol{x}^{n-1})].$$

3. (a) State what is meant by *zero stability* for a multistep method.

   (b) Prove or give a counterexample: every Runge–Kutta method is zero stable.

   (c) Consider the scheme

$$\boldsymbol{x}^{n+1} - (1+\gamma)\boldsymbol{x}^n + \gamma\boldsymbol{x}^{n-1} = \frac{k}{2}\,[(3-\gamma)\boldsymbol{f}(\boldsymbol{x}^n) - (\gamma+1)\boldsymbol{f}(\boldsymbol{x}^{n-1})].$$

   Work out its order (which may depend on $\gamma \in \mathbb{R}$) and determine what values of γ are suitable for use.

4. Let $u = u(x)$ solve the following boundary-value problem

$$\begin{aligned} &u''(x) - 2u(x) = \sin(x) - 1 \quad \text{for } x \in [0,1], \\ &u'(0) - u(0) = 0, \qquad u(1) = 0. \end{aligned}$$

   (a) Assuming that $u \in C^3([0,1])$, write down the Taylor expansion for $u(h)$ centred at 0 with accuracy $O(h^3)$, taking into account that u satisfies the differential equation.

   (b) Based on (a), find a second-order approximation for the boundary condition $u'(0) - u(0) = 0$ in terms of $u(0)$ and $u(h)$. Justify your approximation.

5. For the equation $\partial_t u(x,t) + a\partial_x u(x,t) = 0$ (a is a constant), consider the following approximation scheme

$$\frac{u_m^{n+1} - \frac{1}{2}(u_{m+1}^n + u_{m-1}^n)}{\tau} + a\frac{u_{m+1}^n - u_{m-1}^n}{2h} = 0 \quad \text{with } n \in \mathbb{Z}_+,\ m \in \mathbb{Z}. \qquad (1)$$

   (a) Considering a particular solution $u_m^n = \lambda(\phi)^n e^{im\phi}$, where $\phi \in [0, 2\pi]$, to the problem (1), find the amplification factor $\lambda(\phi)$.

   (b) Now let $\tau = rh/a$ for some constant r. Using the spectral stability test, determine under which conditions on the discretisation parameters $\tau > 0$ and $h > 0$ the scheme (1) is stable.

6. (a) Consider the iteration

$$\boldsymbol{x}^{k+1} = \boldsymbol{G}\boldsymbol{x}^k + \boldsymbol{c}$$

where $\boldsymbol{G}$ is a square matrix, and $\boldsymbol{x}^j$ and $\boldsymbol{c}$ are vectors, with $\boldsymbol{x}^0$ given. Assuming that the equation $\boldsymbol{x} = \boldsymbol{G}\boldsymbol{x} + \boldsymbol{c}$ has a unique solution, state necessary and sufficient conditions on $\boldsymbol{G}$ that guarantee convergence of the iteration.

(b) Given a real matrix

$$\boldsymbol{A} = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \alpha & 0 \\ \beta & 0 & \alpha \end{pmatrix} \quad \text{with } \alpha \neq 0,$$

and some vector $\boldsymbol{b} \in \mathbb{R}^3$, we seek to solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ using the Gauss–Seidel iteration process. Find the transition matrix $\boldsymbol{G}$ corresponding to $\boldsymbol{A}$.

(c) Find all values of parameters α and β such that the Gauss–Seidel iteration process converges for arbitrary initial vector $\boldsymbol{x}^0$.

SECTION B

7. (a) For a numerical scheme $\boldsymbol{x}^{n+1} = \Phi(\boldsymbol{x}^n; k)$, define what is meant by (i) the local truncation error, and (ii) the order of the scheme.

(b) For all $\theta \in [0, 1]$, determine the order of the θ-method,

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + k[\theta \boldsymbol{f}(\boldsymbol{x}^n) + (1 - \theta)\boldsymbol{f}(\boldsymbol{x}^{n+1})]. \tag{2}$$

(c) One can solve (2) using the fixed-point iterations $\boldsymbol{y}_0 = \boldsymbol{x}^n$ and

$$\boldsymbol{y}_{m+1} = \boldsymbol{x}^n + k[\theta \boldsymbol{f}(\boldsymbol{x}^n) + (1 - \theta)\boldsymbol{f}(\boldsymbol{y}_m)].$$

Stating any necessary assumptions, prove that these iterations converge.

(d) Show that if one terminates the fixed-point iterations at $\boldsymbol{y}_2$, one obtains a Runge–Kutta scheme; write down its Butcher table.

8. (a) Define the notions of stability region and A-stability for one-step schemes.

(b) Show that the stability region for the implicit Euler scheme $\boldsymbol{x}^n = \boldsymbol{x}^{n-1} + k\boldsymbol{f}(\boldsymbol{x}^n)$ is $\mathcal{S} = \{z \in \mathbb{C} : |z - 1| > 1\}$.

(c) Let $\boldsymbol{x} = (x_1, \cdots, x_N)$ and consider the system of equations

$$x_j' = -j^2 x_j \qquad \text{for } j \in \{1, \cdots, N\}. \tag{3}$$

If we are to solve (3) using the implicit Euler scheme, what restrictions (if any) on the timestep k are necessary? Justify your answer.

(d) Suppose now that we have a fourth-order scheme whose stability region $\mathcal{S}$ satisfies $\mathcal{S} \cap \mathbb{R} = (-1, 0)$. If this scheme were to be used to solve (3), state any necessary restrictions on the timestep k. Justify your answer.

(e) Show ("from first principles") that, when used to integrate (3), any explicit Runge–Kutta scheme would require a timestep restriction.

9. Let $\bar{\Omega} = [0,1] \times [0,1]$ and $u = u(x,y) \in C^1(\bar{\Omega})$ be such that $u|_{\partial\Omega} = 0$; that is

$$u(0,y) = u(1,y) = 0 \text{ for } y \in [0,1] \quad \text{and} \quad u(x,0) = u(x,1) = 0 \text{ for } x \in [0,1].$$

(a) Prove that

$$|u(x,y)| \le \int_0^1 |\partial_x u(s_1,y)|\, ds_1,\ \forall (x,y) \in \bar{\Omega},$$
$$|u(x,y)| \le \int_0^1 |\partial_y u(x,s_2)|\, ds_2,\ \forall (x,y) \in \bar{\Omega}.$$

Hint: use Newton's formula with respect to x for a fixed y and vice versa.

(b) Prove that

$$\int_0^1 \int_0^1 |u(x,y)|^2\, dxdy \le \int_0^1 \int_0^1 |\partial_x u(x,y)|\, dxdy \int_0^1 \int_0^1 |\partial_y u(x,y)|\, dxdy.$$

Hint: estimate $|u(x,y)|^2$, combining inequalities from (a).

Now let $u_{m,n}$, $m \in \overline{0,M}$, $n \in \overline{0,N}$ be a grid function such that

$$u_{0,n} = u_{M,n} = 0,\ n \in \overline{0,N}, \qquad u_{m,0} = u_{m,N} = 0,\ m \in \overline{0,M}.$$

(c) Prove the discrete analogue of (a), namely

$$|u_{m,n}| \le \sum_{k=1}^{M} |u_{k,n} - u_{k-1,n}|,\ \forall (m,n),\ \text{where } m \in \overline{0,M},\ n \in \overline{0,N};$$
$$|u_{m,n}| \le \sum_{k=1}^{N} |u_{m,k} - u_{m,k-1}|,\ \forall (m,n),\ \text{where } m \in \overline{0,M},\ n \in \overline{0,N}.$$

(d) Prove the discrete analogue of (b), namely

$$\sum_{m=0}^{M} \sum_{n=0}^{N} |u_{m,n}|^2 \le \left(\sum_{n=0}^{N} \sum_{k=1}^{M} |u_{k,n} - u_{k-1,n}| \right) \left(\sum_{m=0}^{M} \sum_{k=1}^{N} |u_{m,k} - u_{m,k-1}| \right).$$

10. Consider the eigenvalue problem for the following difference scheme

$$\frac{u_{k+1} - u_{k-1}}{2h} = -\lambda u_k \quad \text{for } 1 \le k \le N-1 \quad \text{with } hN = 1, \tag{4}$$

$$u_0 = 0, \quad u_N = 0. \tag{5}$$

(a) Find the roots μ_1 and μ_2 of the characteristic polynomial associated to the difference equation (4). Find the values of $\mu_1 + \mu_2$ and $\mu_1\mu_2$.

(b) Under the assumption $\mu_1 = \mu_2 := \mu$, the general solution to the difference equation (4) takes the form

$$u_k = c_1\mu^k + c_2 k\mu^k.$$

Taking into account the boundary conditions (5), find all $\lambda \in \mathbb{C}$ such that the problem (4)–(5) has a nonzero solution.

(c) Under the assumption $\mu_1 \neq \mu_2$, the general solution to the difference scheme takes the form

$$u_k = c_1\mu_1^k + c_2\mu_2^k.$$

Taking into account the boundary conditions (5), find all $\lambda \in \mathbb{C}$ such that problem (4)–(5) has a nonzero solution.

SECTION C

11. (a) Is it possible to construct explicit RKSs using collocations? Justify your answer.

(b) What is the maximum order of an s-stage RKS? How does one attain it?

(c) Prove the following: given smooth $d(t)$ and $g(t)$, with $d(t) = 0$ at the s points of Gaussian quadratures in $[0, k]$, we have

$$\left| \int_0^k g(t)d(t)\, \mathrm{d}t \right| \le Ck^{2s}.$$

You may assume $k < 1$.

(d) Show that no *explicit* s-stage RKS can be of order higher than s. Hint: consider $x' = \lambda x$.

(e) Given the Legendre polynomial $P_2(x) = \frac{1}{2}(3x^2 - 1)$, compute explicitly the corresponding collocation Runge–Kutta scheme (evaluate the integrals, but you may leave the expressions in terms of the nodes) and state its order.