Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH1031-WE01

Title: Discrete Mathematics

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **FOUR** answers. All questions carry the same marks.

Revision:	

ED01/2018

1. (i) (a) How many arrangements are there of the letters SYMMETRY?
(b) How many arrangements have the two Ms adjacent?
(c) How many arrangements have no two adjacent letters the same?

(ii) 11 beads, distinguishable only by colour, are arranged on a ring. Suppose that five of the beads are red, three are blue, two are yellow, and one is green. In how many ways can the beads be arranged?

(iii) (a) State the pigeon-hole principle.
(b) For a permutation σ of $1, 2, 3, 4, 5$, let σ_i denote the ith symbol in the permutation, so for example if $\sigma = 32451$, then $\sigma_5 = 1$. Let

$$f(\sigma) = (3 + \sigma_1 - \sigma_2)\sigma_3 - (7 + \sigma_4)\sigma_5,$$

so for example $f(32451) = 16 - 12 = 4$. Show that there exist (at least) two permutations of $1, 2, 3, 4, 5$ for which the value of f is the same.

2. (i) (a) Find a_n satisfying the recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2}, \quad (n \geq 2),$$

with initial conditions $a_0 = 2$, $a_1 = 2$.
(b) Find a_n satisfying the recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2} - 4n + 2, \quad (n \geq 2),$$

with initial conditions $a_0 = 2$, $a_1 = 2$.

(ii) (a) Let $g(x)$ be the generating function of the sequence (b_n). Show that $xg'(x)$ is the generating function of the sequence (nb_n).
(b) Evaluate $\sum_{n=1}^{\infty} 2^{-n} n^2$.

3. (i) Let c_n denote the number of integer solutions $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = n,$$

with each $x_i \geq 0$.
(a) Derive a formula for c_n via a combinatorial argument.
(b) Write down a generating function for c_n, and express it as compactly as possible.
(c) Evaluate $\sum_{n=0}^{\infty} \binom{n+6}{6} 2^{-n}$.

(ii) Jenny is choosing n of her old toys to send to a jumble sale. She can choose from (identical) dinosaur figures, (identical) miniature trombones, and colouring pens. The pens come in any of five colours, and are otherwise identical. She will choose at least three dinosaurs, at least one trombone, and between one and ten pens of each colour.

Let d_n denote the number of ways in which Jenny can select the n toys.
(a) Write down a generating function for d_n and express it as compactly as possible.
(b) Use your generating function to find d_{28}.

4. (i) Seven-letter words are to be created from some or all of the letters LARCH.
   (a) How many such words are possible?
   (b) How many words have exactly two letters A?
   (c) How many words have at least one R and at least one H?

   (ii) For a non-negative integer n, let $I_n = \{i \in \mathbb{Z} : 1 \le i \le n\}$. Let s_n be the number of subsets of I_n which contain no pair of consecutive integers. Thus $s_0 = 1$ (namely $\emptyset$) and $s_1 = 2$ ($\emptyset$ and $\{1\}$).
   (a) Find and explain a recurrence relation for the s_n.
   (b) Use your recurrence relation to find an expression for the generating function $g(x)$ of the sequence (s_n). (You do *not* have to find the coefficients.)

   (iii) Prove that for any positive integer r,

$$\sum_{k=1}^{n} \binom{k+r}{r+1} = \frac{n}{r+2} \binom{n+r+1}{r+1}, \quad (n \ge 1).$$

5. Consider the set $S = \{a, b, c, d, e\}$ of five distinct elements.

   (a) List all subsets of S of cardinality 2, that is, all subsets containing exactly two elements. Label these subsets V_{mn} with $m, n \in \{a, b, c, d, e\}$ for all subsets $\{m, n\}$. For instance, the subset $\{a, b\}$ is labelled V_{ab}.

   (b) Draw a graph $P(V, E)$ where the subsets $V_{mn}, m, n \in \{a, b, c, d, e\}$ are vertices, and where two vertices are joined with an edge $V_{mn}V_{rs}$ if the corresponding subsets are disjoint ($r, s \in \{a, b, c, d, e\}$).

   (c) Let $G(V', E')$ be a simple connected planar graph. (1) Define the girth of $G(V', E')$. (2) Now assume that $G(V', E')$ is such that $|V'| \ge 5$ and that its girth is 5. Prove that

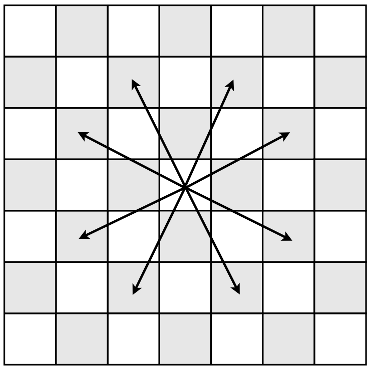
$$|E'| \le \frac{5}{3}(|V'| - 2).$$

   (Hint: you may wish to use the planar handshaking lemma).
   (3) Hence show that the graph $P(V, E)$ constructed in (b) is non planar.

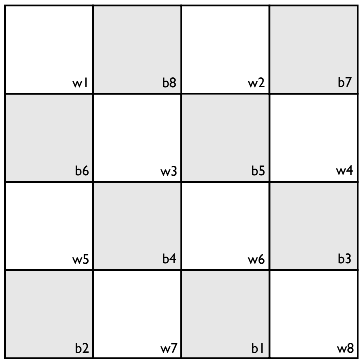
   (d) Given a graph $G(V', E')$, explain what is meant by a subdivision of $G(V', E')$. Since $P(V, E)$ is non planar, show that it contains a subdivision of $K_{3,3}$, using the graph you have drawn in (b).

6. (a) Give the definitions of (1) a cycle or closed path in a graph $G(V, E)$, (2) a bipartite graph.

(b) A Hamiltonian graph is a graph in which there is a cycle passing through every vertex. On a chessboard, a knight always moves two squares in a horizontal or vertical direction and one square in a perpendicular direction. Consider a 4×4



possible knight's moves



4×4 chessboard

chessboard. Draw a graph where each vertex corresponds to a square, and each edge corresponds to a pair of squares connected by a knight's move. Explain why this is a bipartite graph. Is it a complete bipartite graph?

(c) Can a knight visit each square of the 4×4 chessboard just once by a sequence of knight's moves, and finish on the same square as it began? Explain. *(This is called the knight's tour problem on a* 4×4 *chessboard).*

(d) Prove that a bipartite graph with an odd number of vertices is not a Hamiltonian graph.

(e) Use the result proven in (d) to show whether or not a knight's tour on a 5×5 chessboard is possible.