EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH1041-WE01

Title: Programming and Dynamics I

Time Allowed:	2 hours		
Additional Material provided:	None		
Materials Permitted:	None		
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.	
Visiting Students may use dictionaries: No			

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

1. (i) A ball of mass 2 moves along a trajectory $\mathbf{r}(t) = \sin(t)\mathbf{i} + t\sin(t)\mathbf{j}$. Calculate the ball's momentum, acceleration and kinetic energy. By considering the energy of the ball at times $t = n\pi$ where n is an integer, explain why this trajectory does not correspond to a motion in a conservative potential $V(\mathbf{r})$.

   (ii) A particle of mass M moves in one dimension under the influence of a frictional force of magnitude $|v|^\alpha$ where v is its one-dimensional velocity, and α is a constant such that $\alpha < 2$. Initially at $t = 0$, the particle has speed 1. If the particle moves a distance L before coming to rest, calculate α in terms of M and L.

   (iii) A particle of unit mass moves in one dimension so that its position can be written $\mathbf{r}(t) = x(t)\mathbf{i}$ and so that its potential energy is given by

$$V(x) = e^{-x}(x^2 + 4x).$$

   Show that the potential has two equilibrium points, one of which is stable and the other unstable. Determine the frequency of small oscillations around the stable equilibrium point. If the particle is initially at the stable equilibrium point and moving with speed U, for what values of U will $x(t) \to \infty$ as $t \to \infty$?

2. (i) A light spring, with spring constant k, is attached at one end to the ceiling of a room and hangs freely under the influence of gravity. A mass m is attached to the other end and initially supported so that the spring retains its natural length. At time $t = 0$, the support is removed so that mass falls.

   (a) Calculate the time taken for the mass to first return to its original position and the maximum extension of the spring.

   (b) The experiment is repeated, but in this case a shock absorber is also attached to the mass. The shock absorber exerts a force of magnitude $2mb|v|$ where v is the velocity of the mass, and b is a constant such that $0 < b < w = \sqrt{k/m}$. Find $x(t)$ explicitly in this case, and show that the velocity vanishes when $t = \pi/\kappa$ where $\kappa = \sqrt{w^2 - b^2}$. Calculate the maximum extension of the spring in this case.

   (ii) A particle of unit mass has position given as $\mathrm{r} = x\mathbf{i} + z\mathbf{k}$ where $\mathbf{i}$ is a unit vector pointing horizontally and $\mathbf{k}$ is a unit vector pointing vertically upwards. The ground is at $z = 0$ and at time $t = 0$ the particle is fired from the origin with initial velocity $a\mathbf{i} + b\mathbf{k}$. Besides the force of gravity, the particle feels a force from air resistance equal to $-\mathbf{v}$ where $\mathbf{v}$ is the particle's velocity.

   (a) Write down the equations of motion for the particle, and use them to find $x(t)$ and $z(t)$.

   (b) Show that if the particle hits the ground again at time $t = t^*$, it will have travelled a horizontal distance

$$R = \frac{agt^*}{g + b}$$

   where g is the gravitational acceleration.

3. The game of electro-hockey involves a small puck of unit mass and charge which can slide frictionlessly on a horizontal plane. The position of the puck on the plane is given in Cartesian coördinates by $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, and it moves under the influence of a magnetic field $\mathbf{B}$ where $\mathbf{B} = 0$ for $x < 0$ and $\mathbf{B} = B\mathbf{k}$ for $x > 0$, where $B > 0$ is a constant.

   (a) The aim of the game is to hit the puck so that it returns to its original position P whose coördinates are $(-D, 0)$ where $D > 0$. The puck is struck from P with velocity $\mathbf{v} = u\mathbf{i} + (u/2)\mathbf{j}$, where $u > 0$. It first crosses the line $x = 0$ at $t = 0$. What is its position and velocity at this time?

   (b) Write down the Lorentz force law for a charged particle, and use this to deduce the trajectory of the puck $\mathbf{r}(t)$ for $t > 0$ when the puck is in the region $x > 0$.

   (c) Show that the puck will cross the line $x = 0$ again at a time T where

   $$\tan\left(\frac{BT}{2}\right) = -2.$$

   (Hint: It might help to recall that $1 - \cos(\theta) = 2\sin^2(\theta/2)$).

   (d) Show that $\cos(BT) = -3/5$ and $\sin(BT) = -4/5$. It may help to use the relation $1 + \tan^2(\theta) = \sec^2(\theta)$.

   (e) Show that the velocity of the puck when it crosses $x = 0$ at $t = T$ is given by $\mathbf{v} = -u\mathbf{i} + (u/2)\mathbf{j}$, and find its position at this time. Find the value of u in terms of B and D for which the puck returns to the point P.

4. (a) Explain what is meant by a *central force*. Show that angular momentum of a particle subject to a central force is conserved, and that the particle's motion is confined to a plane.

   (b) A planet of unit mass moves in a two-dimensional plane around a fixed star. The planet's position is given in terms of standard polar coördinates (r, θ), with the star located at the origin. At time $t = 0$, the position and velocity of the planet is given in polar coördinates by $\mathbf{r} = 2\mathbf{e}_r$ and $\mathbf{v} = 3\mathbf{e}_r + 4\mathbf{e}_\theta$ respectively. Calculate the planet's angular momentum $\mathbf{L}$.

   (c) If the planet's potential energy is given by $V(r) = -50/r$, show that its total energy E can be written as

   $$E = \frac{\dot{r}^2}{2} + \frac{L^2}{2r^2} - \frac{50}{r}$$

   where $L = |\mathbf{L}|$, and calculate the value of E for the planet with initial conditions given in part (b).

   (d) Find the distance between the planet and the star when the planet is at its closest and its furthest from the star, and calculate the planet's velocity in polar coördinates at each of these points.

   (e) When the planet is closest to the star, its velocity is instantaneously reduced by a factor β so that $\mathbf{v} \to \beta\mathbf{v}$. If the planet subsequently moves in a circular orbit, find β.