Durham University

EXAMINATION PAPER

Examination Session:	**Year:**	**Exam Code:**
May	2018	MATH1051-WE01

Title:
Analysis I

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks.

Revision:	

ED01/2018

1. (a) State what it means for a real sequence $(x_n)_{n\in\mathbb{N}}$ to converge with limit $x^* \in \mathbb{R}$.

   (b) In each of the following cases evaluate $\lim_{n\to\infty} x_n$ or show that it does not exist. Refer to any results you use.

   (i) $x_n = \log(2n+1) - \log(2n-1)$.

   (ii) $x_n = 3^n/n!$.

   (iii) $x_n = \left(n^3 + \log(n)\right)^{1/n}$.

2. (a) Show that any convergent sequence is bounded.

   (b) Give an example of a bounded sequence that is not convergent; justify your answer.

   (c) Suppose that $-C \le x_n \le C$ for all $n \in \mathbb{N}$ and that $(x_n)_{n\in\mathbb{N}}$ is convergent with limit x^*.

   Show that $-C \le x^* \le C$.

   (d) Calculate $\lim_{n\to\infty} x_n$ or show that it does not exist, where:

$$x_n = \frac{1}{\sqrt{1+n^2} - n}.$$

3. (a) Let $(a_n)_{n\in\mathbb{N}}$ be defined by $a_1 = 0$ and $a_{n+1} = a_n^2 + 1/4$.

   (i) Show that $0 \le a_n \le 1/2$ for all $n \in \mathbb{N}$.

   (ii) Show that (a_n) is monotone increasing.

   (iii) Prove that $\lim_{n\to\infty} a_n = 1/2$.

   (b) Let $(b_n)_{n\in\mathbb{N}}$ be defined by $b_1 = 0$ and $b_{n+1} = b_n^2 + 1$. Show that (b_n) is not convergent.

4. (a) Write down

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

   (b) Show

$$\lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right) = 1.$$

   State any result you use.

   (c) Hence or otherwise show that for any positive real number c we have

$$\lim_{x\to 0}\frac{c}{x}\log\left(1+\frac{x}{c}\right) = 1.$$

   State any result you use.

   (d) Evaluate the following limit

$$\log'(c) = \lim_{x\to 0}\frac{\log(c+x) - \log(c)}{x}.$$

5. (a) State the Bolzano-Weierstrass theorem.

   (b) Let $[a,b]$ be a closed and bounded real interval. Prove that a continuous function $f : [a,b] \longrightarrow \mathbb{R}$ is bounded.

   (c) A real number a is defined to be an accumulation point of a sequence $(x_n)_{n\in\mathbb{N}}$ if there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} x_{n_j} = a$.

   Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. Show that the set A of accumulation points of (x_n) is bounded and non-empty.

6. (a) State l'Hôpital's theorem.

   Use l'Hôpital's theorem to show that for each $n \in \mathbb{N}$ we have

$$\lim_{x\to 1} \frac{1-x^n}{1-x} = n.$$

   (b) Give a proof by induction that when $x \neq 1$

$$\frac{1-x^n}{1-x} = \sum_{k=1}^{n} x^{k-1} = 1 + x + x^2 + \cdots + x^{n-1}.$$

   (c) For which positive numbers x does

$$\sum_{k=1}^{\infty} x^{k-1}$$

   converge? Give a proof of your answer.

7. (a) Define what it means for an infinite series to be (i) convergent, (ii) absolutely convergent, (iii) conditionally convergent.

   (b) Show that an absolutely convergent series is convergent.

   (c) Show that the following series is convergent:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+e^{-n}}.$$

   Decide whether it is conditionally convergent or absolutely convergent.

   Justify your answer.

8. Determine whether or not the following series converge. In each case justify your answer:

   (a)
$$\sum_{n=1}^{\infty} \frac{\log(n)}{n^2};$$

   (b)
$$\sum_{n=1}^{\infty} \frac{n^2}{2^n};$$

   (c)
$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)};$$

   (d)
$$\sum_{n=1}^{\infty} \left(\frac{1}{e} + \frac{1}{n}\right)^n.$$

9. (a) Let $[a, b]$ be a real interval and $f : [a, b] \longrightarrow \mathbb{R}$ be a monotone decreasing function. Consider the equidistant partition $\mathcal{P}_n = \{a = x_0, x_1, \ldots, x_n = b\}$ where $x_j = a + j(b-a)/n$.

   Write down the upper and lower Riemann sums $U(f, \mathcal{P}_n)$ and $L(f, \mathcal{P}_n)$.

   Show that f is Riemann integrable.

   (b) Show that
$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \log(2).$$

10. Let I be an interval in $\mathbb{R}$.

   (a) Let $f_n : I \longrightarrow \mathbb{R}$ for $n \in \mathbb{N}$ be a sequence of functions.

   Give the definition of what it means for f_n to converge uniformly to a function $f : I \longrightarrow \mathbb{R}$.

   (b) For $k \in \mathbb{N}$ let $g_k : I \longrightarrow \mathbb{R}$ be a sequence of functions and let $f_n : I \longrightarrow \mathbb{R}$ for $n \in \mathbb{N}$ be defined by
$$f_n(x) = \sum_{k=1}^{n} g_k(x).$$

   State the Weierstrass M-test giving conditions under which f_n converges uniformly on the interval I.

   (c) Prove that the series
$$f_n(x) = \sum_{k=1}^{n} \frac{k\cos(kx)}{x^2 + k^4}$$

   converges uniformly on $\mathbb{R}$. State any result you use.