Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May | 2018 | MATH1061-WE02 |

**Title:** Calculus and Probability I paper 2: Probability

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| Time Allowed: | 1 hour 30 minutes | |
| Additional Material provided: | None | |
| Materials Permitted: | None | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |
| Visiting Students may use dictionaries: No | | |

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| Instructions to Candidates: | Credit will be given for your answers to each question.<br>All questions carry the same marks. |

| Revision: | |
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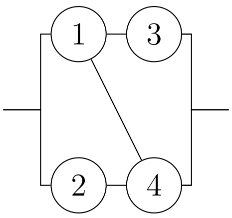
ED01/2018

1. Consider the following reliability network:




Assume that components fail independently, with

$$\mathbb{P}(\text{component } i \text{ works}) = p_i.$$

The answers to the following questions should be given in terms of $p_1, p_2, p_3, p_4$.

(a) What is the probability that the system works, given that component 1 works?

(b) What is the probability that the system works?

(c) What is the probability that component 1 works, given that the system works?

Suppose that components 2 and 4 are no longer independent, but are coupled so that either both work (with probability $p_2$) or both fail.

(d) As a function of $p_1, p_2, p_3$, what is the new probability that component 1 works, given that the system works?

2. (a) The Galactic Insurance Company insures $20,000$ individuals on the planet Mthura. On any one day, the probability that an individual makes a claim is 0.0001; claims from different individuals arise independently. Find, approximately, the probability that on any given day the company receives two or more claims.

(b) If the expected value of an individual claim is 1000 Mthuran grotes, find the expected total value of all claims made in a given year (a year on Mthura lasts exactly 400 days).

(c) The Galactic Insurance Company's analysts re-assess the probability of a claim and estimate the probability that two or more claims are received on a given day to be $1/2$. Using this new data, find, approximately, the probability that two or more claims are received on more than 220 days of a given year (400 days).

In your answers to parts (a)–(c), describe briefly any assumptions that you make in your analysis.

You may use the following values for the cumulative distribution function of the standard normal distribution:

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\Phi(x)$ | 0.5 | 0.841 | 0.977 | 0.999 |

3. (a) A special deck of 12 playing cards consists of the cards Queen, King, Ace of each suit ($\spadesuit, \heartsuit, \diamondsuit, \clubsuit$). You are dealt a hand of 3 cards from this deck, it having been well shuffled.

    i. What is the probability that your hand contains exactly two aces?

    ii. What is the probability that your hand contains exactly two aces, given that you have at least one ace?

   (b) Suppose that the average number of earthquakes (of a certain magnitude measured at a particular monitoring station) is two per year.

    i. Use Markov's inequality to give an upper bound on the probability that at least six earthquakes will occur in the next year.

    ii. Suppose also that the variance of the number of earthquakes per year is equal to 2. Use Chebyshev's inequality to give an upper bound on the probability that at least six earthquakes will occur in the next year.

4. A biased coin has probability 2/3 of landing 'heads', and probability 1/3 of landing 'tails'. The coin is tossed repeatedly until either 'heads' appears for a second time or 'tails' appears for a second time. Let $X$ be the total number of heads obtained, and let $Y$ be the total number of tails obtained. For example, possible sequences of tosses include HH and THT, which give rise to $(X, Y) = (2, 0)$ and $(1, 2)$, respectively.

   (a) Write down in a table the joint distribution of $X$ and $Y$.

   (b) Find the marginal distributions of $X$ and $Y$, and compute $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

   (c) Find $\mathbb{E}(X \mid Y = y)$ for each possible value $y$ of $Y$, and verify that, in this example, $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid Y))$.

   (d) Find $\mathbb{C}\text{ov}(X, Y)$.

5. In your answers to the following questions, you may use, without proof, standard results on moment generating functions, but you should state clearly any result that you use.

   Let $Y$ be Poisson distributed with parameter 1, so that $\mathbb{P}(Y = k) = \mathrm{e}^{-1}/k!$ for $k = 0, 1, 2, \ldots$.

   (a) Compute the moment generating function of $Y$.

   (b) Find $\mathbb{E}(Y^3)$.

   Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with $\mathbb{P}(X_i = 1) = \frac{1}{n}$ and $\mathbb{P}(X_i = 0) = 1 - \frac{1}{n}$. Let $Y_n = X_1 + X_2 + \cdots + X_n$.

   (c) Compute the moment generating function of $X_i$.

   (d) Compute the moment generating function of $Y_n$.

   (e) What can you say about your answer to part (d) in the limit as $n \to \infty$? What does this tell you about $Y_n$ as $n \to \infty$?