Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH1071-WE01

Title:

Linear Algebra I

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for your answers to each question. All questions carry the same marks. Use a separate answer book for each Section.

Revision:	

ED01/2018

SECTION A

Use a separate answer book for this Section.

1. (a) Find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

   (b) Compute the determinant of the matrix

$$\begin{pmatrix} 2 & 1 & 1 & 5 & -3 & 7 \\ 1 & 0 & 1 & 29 & -11 & 23 \\ 4 & 2 & -1 & 17 & -19 & 31 \\ 0 & 0 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{pmatrix}$$

2. (a) Give the equation of the plane Π which contains the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

   in the form $ax + by + cz = d$.

   (b) Find the shortest distance between the origin $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and the plane Π.

3. For each value of $a \in \mathbb{R}$, find the set of solutions to the following system of linear equations.

$$\begin{aligned} 2x + ay + z &= -1, \\ x - y + 3z &= 1, \\ -x + 2y + 2z &= 2. \end{aligned}$$

4. (a) Let $\phi : V \to W$ be a linear map between two vector spaces. Define the *image* of ϕ and show that this is a subspace of W.

   (b) Let $\mathbb{R}[x]_3$ be the vector space of polynomials of degree at most 3 and let $M_2(\mathbb{R})$ be the vector space of 2×2 matrices with real entries. Consider the linear function $\Phi : \mathbb{R}[x]_3 \to M_2(\mathbb{R})$ given by

$$\Phi(f) = \begin{pmatrix} f(0) & f'(0) \\ f(1) & f'(1) \end{pmatrix}$$

   (where we write f' for the first derivative of f).

   Find the rank and nullity of Φ, giving justification for your answer and quoting any theorems that you use.

5. Let $\mathbb{R}[x]_3$ be the vector space consisting of polynomials of degree at most 3. Consider the linear function $T : \mathbb{R}[x]_3 \to \mathbb{R}[x]_3$ given by

$$T(p(x)) = xp'(x) - p(x+1),$$

where we write p' for the first derivative of p.

(a) Write down the matrix of T with respect to the standard basis of $\mathbb{R}[x]_3$.

(b) Determine the rank and nullity of T.

SECTION B

Use a separate answer book for this Section.

6. Let A be the matrix
$$A = \begin{pmatrix} -32 & 24 \\ -40 & 32 \end{pmatrix} .$$
Find a matrix B such that $A = B^3$.

7. Let V be the vector space $\mathbb{R}[x]_3$ of real polynomials of degree at most three and let $\mathcal{L} : V \mapsto V$ be the linear operator
$$\mathcal{L}\left(p(x)\right) = p'(x) + p(x+2)$$
with $p(x) \in \mathbb{R}[x]_3$ and $p'(x) = dp(x)/dx$. Write down the matrix M, representing $\mathcal{L}$ on V, using the standard basis $\{1, x, x^2, x^3\}$, and then find the eigenvalues, the eigenvectors, and the eigenfunctions.

8. Show that
$$(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_1 + x_1y_2 + 2x_2y_2 + x_3y_2 + x_2y_3 + 3x_3y_3$$
defines an inner product on $V = \mathbb{R}^3$ and then find the orthogonal complement to the vector subspace given by
$$U = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$
with respect to this inner product.

9. (i) If A is an $n \times n$ matrix, show that A and its transpose A^t have the same eigenvalues.

   (ii) If B is an $n \times n$ hermitian matrix, show that its eigenvalues are real numbers.

10. Let
$$G = \{I, A, B\}$$
where I denotes the 2×2 identity matrix while
$$A = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} ,$$
$$B = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} .$$
Show that G is a group under matrix multiplication, which you may assume is associative. Present another group of the same order which is isomorphic to G and give explicitly an isomorphism between the two.