Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May | 2018 | MATH1571-WE01 |

**Title:** Single Mathematics B

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| Time Allowed: | 3 hours | |
| Additional Material provided: | Tables: Normal distribution. | |
| Materials Permitted: | None | |
| Calculators Permitted: | Yes | Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS. |
| Visiting Students may use dictionaries: No | | |

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| Instructions to Candidates: | Credit will be given for your answers to each question.<br>All questions carry the same marks. |

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| **Revision:** | |

ED01/2018

1. (a) The polar coordinates of a particle are given by

$$r = t^2, \qquad \theta = \mathrm{e}^{3t}.$$

Find the radial and transverse components of the velocity and acceleration.

(b) i. Find the equations for two lines, one passing through $(1, 2, 3)$ and $(0, 1, 2)$, and the other passing through $(1, 1, 0)$ and $(0, -2, 3)$.

ii. Find the shortest distance between the lines.

(c) Find the volume of the parallelepiped with sides given by $\boldsymbol{a} = 7\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

2. (a) Find the function $y(x)$ which satisfies the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{y}{x} = \mathrm{e}^{x^3}$$

and the condition that $y(1) = 0$.

(b) Find the general solution to the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 8y = \sin(x)\,\mathrm{e}^{3x}.$$

3. A function of period $2\pi$ is defined by

$$f(x) = \begin{cases} 1 & -\pi < x \leq 0, \\ 1 - x & 0 < x \leq \pi. \end{cases}$$

(a) Sketch this function on the interval $(-2\pi, 2\pi)$.

(b) Find the Fourier series for this function.

4. (a) If $f(x,y)$ is a function of $x$ and $y$ where $x = ve^u$ and $y = ve^{-u}$, show that

$$vf_{uv} - f_u = x^2 f_{xx} - y^2 f_{yy}\,,$$

where the suffices indicate partial differentiation. [Hint: assume $f_{xy} = f_{yx}$.]

(b) Find and classify the critical points of the function

$$f(x,y) = x^2 + 3y^2 - y^4 + x^2y^2.$$

(c) The temperature in a lake is given by

$$T(x,y,z) = 10 + z + xy + x^2 + y^2,$$

where the surface is at $z = 0$. A diver at the position $(1,1,-10)$ begins to feel cold and wishes to warm up as quickly as possible. In which direction (written as a unit 3-vector) should he swim? Given that he can swim at a speed $|\boldsymbol{v}| = 1$, what is the rate at which the water temperature increases in this direction?

5. (a) Evaluate the integral

$$\iint_A (x^2 + 2xy)dxdy$$

where

i. $A$ is the rectangle $1 \le x \le 2$, $1 \le y \le 3$;

ii. $A$ is the triangle with vertices $(1,1)$, $(2,1)$ and $(2,3)$.

(b) The curl of two vector fields $\mathbf{V}$ and $\mathbf{W}$ obey the distributivity relation,

$$\nabla \times (\mathbf{V} + \mathbf{W}) = \nabla \times \mathbf{V} + \nabla \times \mathbf{W}.$$

i. Use this fact to evaluate the curl of the vector field

$$\mathbf{U} = (f(x) + y,\ g(y) + z,\ h(z) + x),$$

where $f(x)$, $g(y)$ and $h(z)$ are any arbitrary functions.

ii. Hence or otherwise, determine if the following differential is exact or inexact:

$$dF = (f(x) + y)\,dx + (g(y) + z)\,dy + (h(z) + x)\,dz.$$

6. (a) i. Show that a polynomial function $f(z) = u + iv$ of $z = x + iy$ satisfies the Cauchy-Riemann equations, $u_x = v_y$, and $u_y = -v_x$.

ii. Hence show that both the $u$ and $v$ components of $f(z)$ satisfy Laplace's equation, $\nabla^2 u = \nabla^2 v = 0$.

(b) A complex function is given by the following power series in $z = x + iy$:

$$f(z) = 1 + e^z + e^{2z} + e^{3z} + \ldots$$

i. Using either the root formula test or the ratio test, determine a condition on $x$ that is *a sufficient* condition for the series to be convergent.

ii. Explain your result with reference to the Taylor series of some function, and find a value of $y$ for which this function is defined for *all values of* $x$.

7. (a) A tomato plantation yields fruits of which a proportion $p$ is considered defective, $0 \leq p \leq 1$. The defective fruits are produced at random. The farmer wants to pack these tomatoes in boxes of 20 while keeping the chance of one or more bad tomatoes in the box to be no more than 10%.

   i. What is the maximum acceptable value of $p$ such that this quality benchmark is satisfied?

   ii. Using the value found in item (a), calculate the expected value and standard deviation of the number of defective tomatoes in a box of size 20.

   (b) The same tomato farmer has managed to get the proportion of bad tomatoes produced to $p = 0.02$ this year, and decided to pack them in larger boxes of size 100.

   i. Find the Poisson approximation that a given box will contain $t$ bad tomatoes.

   ii. Suppose a restaurant buys $n$ boxes, $n \geq 0$. Find the expected number and standard deviation of the number of boxes with no bad tomatoes.

   iii. Assume a restaurant orders $n = 50$ boxes. Using a suitable approximation, compute the probability that at most 10 boxes contain bad tomatoes.

   (c) A random variable $X$ is Poisson distributed with parameter $\lambda$, that is,

$$P(X = k) = \frac{exp(-\lambda)\lambda^k}{k!}, \quad k \in \{0, 1, 2, \ldots\}.$$

   i. Show that $E(X) = \lambda$ and $Var(X) = \lambda$.

   ii. Assume that $n$ independent samples are taken from a Poisson with parameter $\lambda$. Find the expected value and variance of the sample mean, that is, $E(\bar{X})$ and $Var(\bar{X})$ where $\bar{X} = \sum_{i=1}^{n} X_i/n$.