

## EXAMINATION PAPER

Examination Session: May

2018

Year:

Exam Code:

MATH2011-WE01

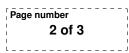
Title:

Complex Analysis II

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

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Revision:



## SECTION A

- 1. (a) Define what it means for a complex valued function f(z) defined on the complex plane to be:
  - (i) complex differentiable at a point  $z_0$ ;
  - (ii) holomorphic at a point  $z_0$ .
  - (b) State the Cauchy-Riemann equations, and use them to find where the function

$$f(x+iy) = x^3 - 3xy^2 + y^3 - 4xy - 3y + i(x^3 + 3x^2y - y^3 - x^2 - 2y^2)$$

is complex-differentiable and where it is holomorphic. State carefully any results from the module that you use.

- 2. (a) Define what is meant by an open ball and an open set in a metric space (X, d). Show that an open ball in a metric space is an open set.
  - (b) Let x and y be two different points in a metric space X. Show that there exist two open *disjoint* sets containing x and y respectively.
- 3. State what it means for a real-valued function defined on  $\mathbb{C}$  to be harmonic. Show that the function  $u(x, y) = 3xy^2 - 4xy^2 - x^3$  is harmonic, and find its harmonic conjugate function v(x, y) (that is to say, a real-valued function v(x, y) such that f(z) = u(x, y) + iv(x, y) is holomorphic).
- 4. (a) State Liouville's theorem.
  - (b) Deduce from Liouville's theorem that if p(z) is a nonconstant polynomial with complex coefficients then there is  $z \in \mathbb{C}$  such that p(z) = 0.
- 5. (a) State Rouché's theorem.
  - (b) Fix R > 0. Prove that if N is sufficiently large, depending on R, then

$$\sum_{k=0}^{N} \frac{z^k}{k!} = 0$$

has no solutions  $z \in D(0, R)$ . You can use any properties of the exponential function that you like, provided they are stated clearly.

- 6. (a) State the complex version of the fundamental theorem of calculus.
  - (b) Using the definition of contour integrals (i.e., without using residues), compute for r > 0

$$\oint_{|z|=r} \frac{1}{z} dz$$

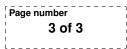
(c) Explain the answer you obtained for

$$\oint_{|z|=r} \frac{1}{z} dz$$

in terms of the residues of  $g(z) = z^{-1}$ .

(d) Deduce from parts (a) and (b) that  $g(z) = z^{-1}$  has no holomorphic antiderivative on any annulus

$$A_r = \{ z \in \mathbb{C} : \frac{r}{2} < |z| < 2r \}, \quad r > 0.$$



## SECTION B

- 7. (a) (i) Find the unique Möbius transformation f(z) which maps the ordered set of points  $\{1, -1, i\}$  to the ordered set of points  $\{\infty, 0, -i\}$ .
  - (ii) Find the image of the region

$$R = \{ z \in \mathbb{C} : |z| < 1, \operatorname{Im}(z) > 0 \}$$

under the map f(z).

- (iii) Show that the map  $g : z \mapsto z^2$  is conformal on the image f(R), then demonstrate that the function  $g \circ f$  defines a conformal map from R to the upper half-plane Im(z) > 0. State any results from the module that you use.
- (b) Prove that any Möbius transformation associated with a matrix from the set  $SL_2(\mathbb{R})$  (that is, the set of real-valued  $2 \times 2$  matrices with unit determinant) maps the upper half-plane Im(z) > 0 to itself.
- 8. (a) (i) Define what it means for a sequence  $\{f_n\}$  of functions to converge pointwise and to converge uniformly on a set X of complex numbers.
  - (ii) Show that the sequence  $\{z^{-n}\}$  converges pointwise, but not uniformly, on |z| > 1. State any results from the module that you use.
  - (b) State the Weierstrass M-test. Given  $0 < r < R < \infty$ , show that the series

$$\sum_{n=1}^{\infty} \frac{(z+\frac{1}{z})^n}{n!}$$

converges uniformly on  $\{z \in \mathbb{C} : r < |z| < R\}$ . Thus, show that the series converges on the punctured complex plane  $z \neq 0$  to a continuous function. Argue carefully and give a statement of any result you use.

- 9. (a) State Cauchy's Residue Theorem for simple closed curves.
  - (b) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{x^2 + 1} dx,$$

stating clearly any results that you use.

- 10. Let D = D(0, 1) be the open unit disc in the complex plane and suppose that f is holomorphic on D with f(0) = 0 and  $|f(z)| \le 1$  for all  $z \in D$ .
  - (a) Explain why  $g(z) = \frac{f(z)}{z}$  can be extended to a well defined holomorphic function on D.
  - (b) For each 0 < r < 1, prove  $|g(z)| \le \frac{1}{r}$  for  $z \in D(0, r)$ .
  - (c) Prove that  $|g(z)| \leq 1$  for all  $z \in D$  and deduce that for all  $z \in D$ ,  $|f(z)| \leq |z|$ .
  - (d) Show that if either |f'(0)| = 1 or |f(z)| = |z| for some nonzero  $z \in D$ , then f(z) = az for all  $z \in D$ , for some constant a with |a| = 1.