Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May | 2018 | MATH2011-WE01 |

**Title:**

Complex Analysis II

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| Time Allowed: | 3 hours | |
| Additional Material provided: | None | |
| Materials Permitted: | None | |
| Calculators Permitted: | No | Models Permitted: Use of electronic calculators is forbidden. |
| Visiting Students may use dictionaries: No | | |

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| Instructions to Candidates: | Credit will be given for:<br>the best **FOUR** answers from Section A<br>and the best **THREE** answers from Section B.<br>Questions in Section B carry **TWICE** as many marks as those in Section A. |

| Revision: | |
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ED01/2018

## SECTION A

1. (a) Define what it means for a complex valued function $f(z)$ defined on the complex plane to be:

   (i) complex differentiable at a point $z_0$;

   (ii) holomorphic at a point $z_0$.

   (b) State the Cauchy-Riemann equations, and use them to find where the function

$$f(x+iy) = x^3 - 3xy^2 + y^3 - 4xy - 3y + i(x^3 + 3x^2y - y^3 - x^2 - 2y^2)$$

   is complex-differentiable and where it is holomorphic. State carefully any results from the module that you use.

2. (a) Define what is meant by an open ball and an open set in a metric space $(X, d)$. Show that an open ball in a metric space is an open set.

   (b) Let $x$ and $y$ be two different points in a metric space $X$. Show that there exist two open *disjoint* sets containing $x$ and $y$ respectively.

3. State what it means for a real-valued function defined on $\mathbb{C}$ to be harmonic. Show that the function $u(x, y) = 3xy^2 - 4xy^2 - x^3$ is harmonic, and find its harmonic conjugate function $v(x, y)$ (that is to say, a real-valued function $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is holomorphic).

4. (a) State Liouville's theorem.

   (b) Deduce from Liouville's theorem that if $p(z)$ is a nonconstant polynomial with complex coefficients then there is $z \in \mathbb{C}$ such that $p(z) = 0$.

5. (a) State Rouché's theorem.

   (b) Fix $R > 0$. Prove that if $N$ is sufficiently large, depending on $R$, then

$$\sum_{k=0}^{N} \frac{z^k}{k!} = 0$$

   has no solutions $z \in D(0, R)$. You can use any properties of the exponential function that you like, provided they are stated clearly.

6. (a) State the complex version of the fundamental theorem of calculus.

   (b) Using the definition of contour integrals (i.e., without using residues), compute for $r > 0$

$$\oint_{|z|=r} \frac{1}{z} dz.$$

   (c) Explain the answer you obtained for

$$\oint_{|z|=r} \frac{1}{z} dz$$

   in terms of the residues of $g(z) = z^{-1}$.

   (d) Deduce from parts (a) and (b) that $g(z) = z^{-1}$ has no holomorphic antiderivative on any annulus

$$A_r = \{z \in \mathbb{C} : \frac{r}{2} < |z| < 2r\}, \quad r > 0.$$

## SECTION B

7. (a) (i) Find the unique Möbius transformation $f(z)$ which maps the ordered set of points $\{1, -1, i\}$ to the ordered set of points $\{\infty, 0, -i\}$.

   (ii) Find the image of the region

$$R = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im}(z) > 0\}$$

   under the map $f(z)$.

   (iii) Show that the map $g : z \mapsto z^2$ is conformal on the image $f(R)$, then demonstrate that the function $g \circ f$ defines a conformal map from $R$ to the upper half-plane $\operatorname{Im}(z) > 0$. State any results from the module that you use.

   (b) Prove that any Möbius transformation associated with a matrix from the set $\mathrm{SL}_2(\mathbb{R})$ (that is, the set of real-valued $2 \times 2$ matrices with unit determinant) maps the upper half-plane $\operatorname{Im}(z) > 0$ to itself.

8. (a) (i) Define what it means for a sequence $\{f_n\}$ of functions to converge pointwise and to converge uniformly on a set $X$ of complex numbers.

   (ii) Show that the sequence $\{z^{-n}\}$ converges pointwise, but not uniformly, on $|z| > 1$. State any results from the module that you use.

   (b) State the Weierstrass M-test. Given $0 < r < R < \infty$, show that the series

$$\sum_{n=1}^{\infty} \frac{(z + \frac{1}{z})^n}{n!}$$

   converges uniformly on $\{z \in \mathbb{C} : r < |z| < R\}$. Thus, show that the series converges on the punctured complex plane $z \neq 0$ to a continuous function. Argue carefully and give a statement of any result you use.

9. (a) State Cauchy's Residue Theorem for simple closed curves.

   (b) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{x^2 + 1} dx,$$

   stating clearly any results that you use.

10. Let $D = D(0, 1)$ be the open unit disc in the complex plane and suppose that $f$ is holomorphic on $D$ with $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D$.

   (a) Explain why $g(z) = \frac{f(z)}{z}$ can be extended to a well defined holomorphic function on $D$.

   (b) For each $0 < r < 1$, prove $|g(z)| \leq \frac{1}{r}$ for $z \in D(0, r)$.

   (c) Prove that $|g(z)| \leq 1$ for all $z \in D$ and deduce that for all $z \in D$, $|f(z)| \leq |z|$.

   (d) Show that if either $|f'(0)| = 1$ or $|f(z)| = |z|$ for some nonzero $z \in D$, then $f(z) = az$ for all $z \in D$, for some constant $a$ with $|a| = 1$.