EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH2031-WE01

Title:
Analysis in Many Variables II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. (a) Compute the gradient, ∇f, of $f(x,y,z) = (x^2+xy)/(2+z)$. In which direction is this function increasing the fastest at the point $(1,3,-1)$, and what is the equation for the tangent plane to the surface $f(x,y,z) = 4$ at this point?

   (b) Let $F(t)$ be the value of $f(x,y,z)$ restricted to the curve $x = t^2$, $y = t^3$, $z = 2t$. Use the chain rule to calculate $\frac{dF}{dt}$.

2. (a) (i) Sketch the two-dimensional vector field $\mathbf{A}(x,y) = \frac{1}{2}y\mathbf{e}_1 - \frac{1}{2}x\mathbf{e}_2$.

   (ii) Compute the divergence of $\mathbf{A}$ and comment on its value in relation to your sketch.

   (b) Consider the transformation from polar coordinates (r,θ) back to cartesian coordinates (x,y):

$$\mathbf{x}(r,\theta) \quad = \quad \begin{pmatrix} x(r,\theta) \\ y(r,\theta) \end{pmatrix} \quad = \quad \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

   By calculating the Jacobian $J(\mathbf{x})$, show whether the transformation is orientation preserving or not.

3. (a) If $\mathbf{v}(\mathbf{x})$ is a general vector field in 3 dimensions, give an expression for $(\nabla \times \mathbf{v})_i$ the i^{th} component of the curl of $\mathbf{v}$, in index notation.

   (b) Write down an identity relating the product $\varepsilon_{ijk}\varepsilon_{klm}$ of two Levi-Civita ε symbols to a combination of products of Kronecker delta symbols.

   (c) Using index notation, calculate $\nabla\cdot(\mathbf{x}\times(\mathbf{x}\times\mathbf{a}))$, where $\mathbf{x}$ and $\mathbf{a}$ are 3-dimensional position and constant vectors respectively.

4. (a) Compute the double integral

$$I = \int_0^1 \int_x^{x+1} xy^2 \, dy \, dx \, .$$

   (b) Next, change the order of integration and re-evaluate the integral.

   (c) Give the name of the theorem which states that your results from parts (a) and (b) should agree.

5. (a) State the divergence theorem.

   (b) A unit cube sits on the $x-y$ plane, so that the set of points inside the cube is

$$V = \{(x,y,z) \in \mathbb{R}^3 \,:\, |x| < 1/2,\, |y| < 1/2,\, 0 < z < 1\}\,.$$

   Using the divergence theorem, compute $\int_{S_v} \mathbf{F}\cdot d\mathbf{A}$, where

$$\mathbf{F}(x,y,z) = x\,\mathbf{e}_1 + 2y\,\mathbf{e}_2 + 3(z^2-1)\,\mathbf{e}_3$$

   and the surface S_v comprises the four *vertical* faces of the cube, that is to say the entire surface S of the cube minus the two faces parallel to the $x-y$ plane.

6. Find the coefficients a, b, c, d and g in the following generalised function identities:

   (a) $e^{2x}\delta(x-1) = a\,\delta(x-1)\,;$

   (b) $e^{2x}\delta'(x-1) = b\,\delta'(x-1) + c\,\delta(x-1)\,;$

   (c) $\delta(2x-1) = d\,\delta(x-g)\,.$

SECTION B

7. (a) Find and classify all the critical points of

$$f(x,y) \;=\; (x^2-y)e^{-(x^2+y^2)}.$$

   (b) Find the maxima and minima of $f(x,y)$ on the circle $x^2+y^2 = 1/2$ by EITHER

      (i) using a more appropriate coordinate system, OR

      (ii) using Lagrange multipliers in the original (x,y) coordinate system.

   (c) Hence find the global maximum and minimum of $f(x,y)$ over the disc D defined as $\{D : x^2+y^2 \leq 1/2\}$, stating any results you use about the location of global extremum.

   (d) Sketch a plot of D and include all the points of interest you have found.

8. (a) Consider a vector field $\mathbf{v}(\mathbf{x}) : U \to \mathbb{R}^n$, with U open in $\mathbb{R}^n$. Give the definition of $\mathbf{v}$ being differentiable at a point $\mathbf{a} \in U$. Your definition should include the explicit form of the linear function $\mathbf{L}(\mathbf{h})$.

   (b) Prove that the differential (i.e. the Jacobian Matrix) of the composite vector field $\mathbf{u}(\mathbf{x}) = \mathbf{w}(\mathbf{v}(\mathbf{x}))$ is given by

$$D\mathbf{u}(\mathbf{x}) = D\mathbf{w}(\mathbf{v})D\mathbf{v}(\mathbf{x})$$

   where $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$ are all differentiable vector fields in $\mathbb{R}^n$.

   (c) Calculate the differentials $D\mathbf{v}(\mathbf{x})$ and $D\mathbf{w}(\mathbf{x})$ of the transformations $\mathbf{v}(\mathbf{x})$ and $\mathbf{w}(\mathbf{x})$ given by

$$\mathbf{v}(\mathbf{x}) \;=\; \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \;=\; \begin{pmatrix} \cos y \\ \sin x \end{pmatrix} \qquad \text{and} \qquad \mathbf{w}(\mathbf{x}) \;=\; \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \;=\; \begin{pmatrix} x^2-y^2 \\ 2xy \end{pmatrix}.$$

   (d) Use the results of part (b) and (c) to find the Jacobian $J(\mathbf{u})$ of the composite transformation $\mathbf{u}(\mathbf{x}) = \mathbf{w}(\mathbf{v}(\mathbf{x}))$, *without explicitly evaluating* $D\mathbf{u}(\mathbf{x})$, where $\mathbf{v}(\mathbf{x})$ and $\mathbf{w}(\mathbf{x})$ are defined as in part (c).

9. (a) State Stokes' theorem.

(b) A surface S, forming half of the curved surface of a unit-length and unit-radius cylinder, is given in parametric form as

$$S : (u, v) \mapsto \mathbf{x}(u, v) = \cos v\, \mathbf{e}_1 + u\, \mathbf{e}_2 + \sin v\, \mathbf{e}_3\,, \quad 0 \le u \le 1\,,\ 0 \le v \le \pi\,.$$

Taking $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ to point along the x, y and z axes respectively, sketch S.

(c) For a point $\mathbf{x}(u, v)$ on S, compute

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\,.$$

What is the geometrical interpretation of this vector?

(d) Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = -yz\, \mathbf{e}_1 + x\, \mathbf{e}_2 + xy\, \mathbf{e}_3\,.$$

Using your result from part (c), compute the surface integral $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$, where S is the surface defined in part (b).

(e) The boundary of S consists of two straight-line segments and two semicircles. Compute the line integrals $\int \mathbf{F} \cdot d\mathbf{x}$ along each of these pieces, and show that their sum is consistent with Stokes' theorem, given your answer to part (d).

10. For $0 < x < 1$ and $t > 0$, the temperature $T(x, t)$ along a metal bar of unit length satisfies

$$\frac{\partial T(x, t)}{\partial t} = \frac{\partial^2 T(x, t)}{\partial x^2}\,.$$

The end of the bar at $x = 0$ is insulated, while the end of the bar at $x = 1$ is held at zero temperature, leading to the boundary conditions

$$\frac{\partial T(0, t)}{\partial x} = 0\,, \quad T(1, t) = 0 \qquad \text{for } t > 0\,.$$

(a) Use the method of separation of variables to find a set of solutions to this problem of the form $T(x, t) = X(x)Y(t)$ where X and Y are real functions, taking care to make sure X is consistent with the stated boundary conditions.

(b) Then use a superposition of these solutions to find $T(x, t)$ for all $t > 0$ if the temperature distribution at $t = 0$ is

$$T(x, 0) = T_0(x) = 1 \quad \text{for } 0 < x < 1\,.$$

(Hint/reminder: to find the coefficients in your general superposition, try multiplying by one of the separated solutions from part (a) and integrating from 0 to 1. The formula $\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A + B) + \cos(A - B)\right)$ can be used without proof.)

(c) Show that as $t \to \infty$ the temperature at the left-hand end of the bar, $T(0, t)$, is given approximately by $T(0, t) \approx \alpha \exp(\beta t)$, where α and β are two constants which you should determine.