Durham University

# EXAMINATION PAPER

| Examination Session: | Year: | Exam Code: |
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| May | 2018 | MATH2051-WE01 |

| Title: |
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| Numerical Analysis II |

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| Time Allowed: | 3 hours | |
| Additional Material provided: | None | |
| Materials Permitted: | None | |
| Calculators Permitted: | Yes | Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS. |
| Visiting Students may use dictionaries: No | | |

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| Instructions to Candidates: | Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A. |

| Revision: | |
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ED01/2018

## SECTION A

1. Let $F$ be a floating-point number system with base 10, exponent in $\{-1, 0, 1\}$ and fraction of the form $\pm(0.d_1d_2)$, where $d_1 \neq 0$.

   (a) Suppose we need to compute the $\ell_2$-norm of the vector $\boldsymbol{x} = (1.6, 1.7, 1.8, 1.9)^\top$ in $\mathbb{R}^4$. Find the exact answer $\|\boldsymbol{x}\|_2$, and round it to the nearest number in $F$.

   (b) If we try to compute $\|\boldsymbol{x}\|_2$ in floating-point arithmetic using the formula

   $$\left(\sum_{k=1}^{4} x_k^2\right)^{1/2},$$

   rounding to the nearest number in $F$ at each stage of the calculation, what will go wrong?

   (c) Show how we could compute the correctly rounded value using the same number system $F$ and the same rounding.

2. The Hermite interpolating polynomial for a smooth function $f : \mathbb{R} \to \mathbb{R}$ on a set of nodes $x_0 < x_1 < \ldots < x_n$ takes the form

   $$p_{2n+1}(x) = \sum_{k=0}^{n} \Big( f(x_k)h_k(x) + f'(x_k)\hat{h}_k(x) \Big)$$

   where the basis functions $h_k$ and $\hat{h}_k$ have the form

   $$h_k(x) = \ell_k^2(x)\Big(1 - 2(x - x_k)\ell_k'(x_k)\Big), \quad \hat{h}_k(x) = \ell_k^2(x)(x - x_k)$$

   for suitable functions $\ell_k(x)$.

   For $x \in [x_0, x_n]$, prove that the interpolation error satisfies

   $$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{i=0}^{n} (x - x_i)^2$$

   for some $\xi \in (x_0, x_n)$.

3. (a) Prove that the equation

   $$5x + \ln\big(x + 6012\big) = 10\,095$$

   has exactly one positive root.

   (b) Use Newton's method to find this root, rounded to exactly 4 decimal places.

4. Recall that the conjugate gradient method is a line search method for solving a linear system $A\boldsymbol{x} = \boldsymbol{b}$, where

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k \boldsymbol{d}_k, \quad \alpha_k = -\frac{\boldsymbol{d}_k^\top \boldsymbol{r}_k}{\boldsymbol{d}_k^\top A \boldsymbol{d}_k}, \quad \text{and} \quad \boldsymbol{d}_{k+1} = -\boldsymbol{r}_{k+1} + \beta_k \boldsymbol{d}_k.$$

The matrix $A$ is assumed to be symmetric positive definite.

(a) Define $\boldsymbol{r}_k$ and state its usual name.

(b) By imposing that the new search direction is $A$-conjugate to the previous one, show that (assuming $\boldsymbol{d}_k \neq 0$) we have

$$\beta_k = \frac{\boldsymbol{r}_{k+1}^\top A \boldsymbol{d}_k}{\boldsymbol{d}_k^\top A \boldsymbol{d}_k}.$$

(c) Starting with $\boldsymbol{x}_0 = (0,0)^\top$ and $\boldsymbol{d}_0 = -\boldsymbol{r}_0$, use this method to find the exact solution to the linear system

$$\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

5. (a) Write down the over-determined system of equations that results from trying to fit the data $(0,0)$, $(1,2)$, $(2,5)$ with a linear polynomial $p_1(x) = c_0 + c_1 x$.

(b) Use QR decomposition to find the least-squares solution to this linear system.

6. (a) Find the coefficients $a_0$, $a_1$ in the open Newton-Cotes formula

$$I_1(f) = a_0 f(x_0) + a_1 f(x_1), \quad \text{where} \quad x_0 = \frac{2a+b}{3}, \quad x_1 = \frac{a+2b}{3},$$

and $a, b$ are the integration limits with $a < b$.

(b) What is meant by the "degree of exactness" of a quadrature formula?

(c) Find the degree of exactness of the formula in part (a).

## SECTION B

7. This question concerns the finite-difference approximation

$$f'(x_0) \approx D_n = a_0 f(x_0) + a_1 f(x_1) + \ldots + a_n f(x_n),$$

derived by differentiating the interpolating polynomial $p_n$ for $f$ at the Chebyshev nodes

$$x_j = \cos\left[\frac{(2j+1)\pi}{2(n+1)}\right] \quad \text{for } j = 0, \ldots, n.$$

(a) For the case $n = 2$, write down the nodes $x_0$, $x_1$, $x_2$ explicitly, and find the interpolating polynomial $p_2$ in Lagrange form.

(b) Hence find the coefficients $a_0$, $a_1$ and $a_2$ in the formula $D_2$.

(c) Write down a formula for the interpolation error $f - p_2$, and explain the significance of the Chebyshev nodes in this context.

(d) Use part (c) to find an expression for the truncation error in $D_2$.

(e) In the formula $D_n$ for general $n$, show that

$$a_0 = \frac{T''_{n+1}(x_0)}{2T'_{n+1}(x_0)},$$

where $T_{n+1}(x)$ is the Chebyshev polynomial of degree $n+1$.

[*Hint:* You may assume without proof that the Lagrange basis functions may be written

$$\ell_k(x) = \frac{w_{n+1}(x)}{w'_{n+1}(x_k)(x - x_k)}, \qquad \text{where} \quad w_{n+1}(x) = \prod_{j=0}^{n}(x - x_j).]$$

8. Assume that the function $f(x)$ has a root $x_*$ in the interval $(1, 2)$, and further that $|f'(x)| > 3$ and $|f''(x)| < 4$ for all $x \in (1, 2)$. In addition, let $x_0 \in (1, 2)$ and suppose that Newton's method converges for all such $x_0$.

(a) Write down the iteration function $x_{k+1} = g(x_k)$ for Newton's method, show that $g'(x_*) = 0$, and calculate $g''(x_*)$.

(b) Show that

$$x_* - x_{k+1} = -\frac{f''(x_*)}{2f'(x_*)}(x_* - x_k)^2 + \frac{g'''(\xi_k)}{6}(x_* - x_k)^3$$

for some $\xi_k \in (1, 2)$.

(c) Define superlinear convergence, and use part (b) to prove that Newton's method converges superlinearly in this case.

(d) Using part (b) but ignoring the $g'''(\xi_k)$ term, estimate the number of iterations required for $|x_* - x_k| \leq 10^{-100}$.

9. (a) Prove that the function

$$\|\boldsymbol{x}\|_* = \frac{1}{n}\sum_{j=1}^{n}|x_j| \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n$$

is a vector norm.

(b) For $n = 2$, sketch the set $\left\{\boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \|\boldsymbol{x}\|_* = 1\right\}$.

(c) Define what is meant by an induced matrix norm, and find a simple formula for the matrix norm induced by the vector norm $\|\cdot\|_*$.

(d) Compute this induced norm of the matrix

$$A = \begin{pmatrix} 2^0 & 2^0 & \dots & 2^0 \\ 2^1 & 2^1 & \dots & 2^1 \\ 2^2 & 2^2 & \dots & 2^2 \\ \vdots & \vdots & & \vdots \\ 2^m & 2^m & \dots & 2^m \end{pmatrix}.$$

(e) Show that there are an infinite number of different vector norms that induce this same matrix norm.

10. Let the polynomials $\{\phi_k\}$ be generated by the three-term recurrence relation

$$\phi_{k+1}(x) = x\phi_k(x) - \frac{(x\phi_k, \phi_k)}{(\phi_k, \phi_k)}\phi_k(x) - \frac{(\phi_k, \phi_k)}{(\phi_{k-1}, \phi_{k-1})}\phi_{k-1}(x) \quad \text{for } k > 1,$$

where

$$\phi_0(x) = 1, \quad \phi_1(x) = x - \frac{(x, 1)}{(1, 1)},$$

and the inner product is defined by

$$(\phi_i, \phi_j) = \int_{-1}^{1}(1 - x^2)\phi_i(x)\phi_j(x)\,\mathrm{d}x.$$

(a) Show that $\phi_k$ is an odd (even) function of $x$ if $k$ is odd (even) respectively.

(b) Compute $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$.

(c) Hence derive the nodes and weights for a 3-point Gaussian quadrature formula $\mathcal{G}_{2,w}(f)$ for integrals of the form

$$\int_{-1}^{1}(1 - x^2)f(x)\,\mathrm{d}x.$$

(d) For which of the following functions (if any) will the formula $\mathcal{G}_{2,w}(f)$ give the exact integral? Justify your answer.

(i) $f(x) = x^5 + x^4 + x^3$, (ii) $f(x) = x^6 + x^5 + x^4$, (iii) $f(x) = x^7 + x^5 + x^4$.