

# **EXAMINATION PAPER**

Examination Session: May

2018

Year:

Exam Code:

MATH2051-WE01

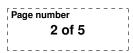
### Title:

# Numerical Analysis II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates: Credit will be given for: the best <b>FOUR</b> answers from Section A and the best <b>THREE</b> answers from Section B. Questions in Section B carry <b>TWICE</b> as many marks as those in Section A.
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Revision:



#### SECTION A

- 1. Let F be a floating-point number system with base 10, exponent in  $\{-1, 0, 1\}$  and fraction of the form  $\pm (0.d_1d_2)$ , where  $d_1 \neq 0$ .
  - (a) Suppose we need to compute the  $\ell_2$ -norm of the vector  $\boldsymbol{x} = (1.6, 1.7, 1.8, 1.9)^{\top}$ in  $\mathbb{R}^4$ . Find the exact answer  $\|\boldsymbol{x}\|_2$ , and round it to the nearest number in F.
  - (b) If we try to compute  $\|\boldsymbol{x}\|_2$  in floating-point arithmetic using the formula

$$\left(\sum_{k=1}^4 x_k^2\right)^{1/2},$$

rounding to the nearest number in F at each stage of the calculation, what will go wrong?

- (c) Show how we could compute the correctly rounded value using the same number system F and the same rounding.
- 2. The Hermite interpolating polynomial for a smooth function  $f : \mathbb{R} \to \mathbb{R}$  on a set of nodes  $x_0 < x_1 < \ldots < x_n$  takes the form

$$p_{2n+1}(x) = \sum_{k=0}^{n} \left( f(x_k)h_k(x) + f'(x_k)\hat{h}_k(x) \right)$$

where the basis functions  $h_k$  and  $\hat{h}_k$  have the form

$$h_k(x) = \ell_k^2(x) \Big( 1 - 2(x - x_k) \ell_k'(x_k) \Big), \quad \hat{h}_k(x) = \ell_k^2(x) (x - x_k)$$

for suitable functions  $\ell_k(x)$ .

For  $x \in [x_0, x_n]$ , prove that the interpolation error satisfies

$$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{i=0}^{n} (x - x_i)^2$$

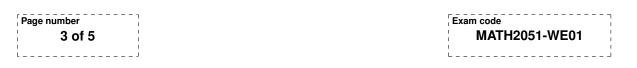
for some  $\xi \in (x_0, x_n)$ .

3. (a) Prove that the equation

$$5x + \ln(x + 6012) = 10\,095$$

has exactly one positive root.

(b) Use Newton's method to find this root, rounded to exactly 4 decimal places.



4. Recall that the conjugate gradient method is a line search method for solving a linear system  $A\mathbf{x} = \mathbf{b}$ , where

$$oldsymbol{x}_{k+1} = oldsymbol{x}_k + lpha_k oldsymbol{d}_k, \quad lpha_k = -rac{oldsymbol{d}_k^ op oldsymbol{r}_k}{oldsymbol{d}_k^ op Aoldsymbol{d}_k}, \quad ext{and} \quad oldsymbol{d}_{k+1} = -oldsymbol{r}_{k+1} + eta_k oldsymbol{d}_k.$$

The matrix A is assumed to be symmetric positive definite.

- (a) Define  $\boldsymbol{r}_k$  and state its usual name.
- (b) By imposing that the new search direction is A-conjugate to the previous one, show that (assuming  $d_k \neq 0$ ) we have

$$\beta_k = \frac{\boldsymbol{r}_{k+1}^\top A \boldsymbol{d}_k}{\boldsymbol{d}_k^\top A \boldsymbol{d}_k}.$$

(c) Starting with  $\boldsymbol{x}_0 = (0,0)^{\top}$  and  $\boldsymbol{d}_0 = -\boldsymbol{r}_0$ , use this method to find the exact solution to the linear system

$$\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

- 5. (a) Write down the over-determined system of equations that results from trying to fit the data (0,0), (1,2), (2,5) with a linear polynomial  $p_1(x) = c_0 + c_1 x$ .
  - (b) Use QR decomposition to find the least-squares solution to this linear system.
- 6. (a) Find the coefficients  $a_0$ ,  $a_1$  in the open Newton-Cotes formula

$$I_1(f) = a_0 f(x_0) + a_1 f(x_1)$$
, where  $x_0 = \frac{2a+b}{3}$ ,  $x_1 = \frac{a+2b}{3}$ ,

and a, b are the integration limits with a < b.

- (b) What is meant by the "degree of exactness" of a quadrature formula?
- (c) Find the degree of exactness of the formula in part (a).

#### SECTION B

7. This question concerns the finite-difference approximation

$$f'(x_0) \approx D_n = a_0 f(x_0) + a_1 f(x_1) + \ldots + a_n f(x_n),$$

derived by differentiating the interpolating polynomial  $p_n$  for f at the Chebyshev nodes

$$x_j = \cos\left[\frac{(2j+1)\pi}{2(n+1)}\right]$$
 for  $j = 0, ..., n$ .

- (a) For the case n = 2, write down the nodes  $x_0$ ,  $x_1$ ,  $x_2$  explicitly, and find the interpolating polynomial  $p_2$  in Lagrange form.
- (b) Hence find the coefficients  $a_0$ ,  $a_1$  and  $a_2$  in the formula  $D_2$ .
- (c) Write down a formula for the interpolation error  $f p_2$ , and explain the significance of the Chebyshev nodes in this context.
- (d) Use part (c) to find an expression for the truncation error in  $D_2$ .
- (e) In the formula  $D_n$  for general n, show that

$$a_0 = \frac{T_{n+1}''(x_0)}{2T_{n+1}'(x_0)}$$

where  $T_{n+1}(x)$  is the Chebyshev polynomial of degree n+1.

 $[\mathit{Hint:}$  You may assume without proof that the Lagrange basis functions may be written

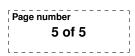
$$\ell_k(x) = \frac{w_{n+1}(x)}{w'_{n+1}(x_k)(x-x_k)}, \quad \text{where} \quad w_{n+1}(x) = \prod_{j=0}^n (x-x_j).$$

- 8. Assume that the function f(x) has a root  $x_*$  in the interval (1, 2), and further that |f'(x)| > 3 and |f''(x)| < 4 for all  $x \in (1, 2)$ . In addition, let  $x_0 \in (1, 2)$  and suppose that Newton's method converges for all such  $x_0$ .
  - (a) Write down the iteration function  $x_{k+1} = g(x_k)$  for Newton's method, show that  $g'(x_*) = 0$ , and calculate  $g''(x_*)$ .
  - (b) Show that

$$x_* - x_{k+1} = -\frac{f''(x_*)}{2f'(x_*)}(x_* - x_k)^2 + \frac{g'''(\xi_k)}{6}(x_* - x_k)^3$$

for some  $\xi_k \in (1, 2)$ .

- (c) Define superlinear convergence, and use part (b) to prove that Newton's method converges superlinearly in this case.
- (d) Using part (b) but ignoring the  $g'''(\xi_k)$  term, estimate the number of iterations required for  $|x_* x_k| \le 10^{-100}$ .



9. (a) Prove that the function

$$\|\boldsymbol{x}\|_* = \frac{1}{n} \sum_{j=1}^n |x_j| \text{ for } \boldsymbol{x} \in \mathbb{R}^n$$

is a vector norm.

- (b) For n = 2, sketch the set  $\{ \boldsymbol{x} \in \mathbb{R}^n \mid ||\boldsymbol{x}||_* = 1 \}$ .
- (c) Define what is meant by an induced matrix norm, and find a simple formula for the matrix norm induced by the vector norm  $\|\cdot\|_*$ .
- (d) Compute this induced norm of the matrix

$$A = \begin{pmatrix} 2^0 & 2^0 & \dots & 2^0 \\ 2^1 & 2^1 & \dots & 2^1 \\ 2^2 & 2^2 & \dots & 2^2 \\ \vdots & \vdots & & \vdots \\ 2^m & 2^m & \dots & 2^m \end{pmatrix}$$

- (e) Show that there are an infinite number of different vector norms that induce this same matrix norm.
- 10. Let the polynomials  $\{\phi_k\}$  be generated by the three-term recurrence relation

$$\phi_{k+1}(x) = x\phi_k(x) - \frac{(x\phi_k, \phi_k)}{(\phi_k, \phi_k)}\phi_k(x) - \frac{(\phi_k, \phi_k)}{(\phi_{k-1}, \phi_{k-1})}\phi_{k-1}(x) \quad \text{for } k > 1,$$

where

$$\phi_0(x) = 1, \quad \phi_1(x) = x - \frac{(x,1)}{(1,1)},$$

and the inner product is defined by

$$(\phi_i, \phi_j) = \int_{-1}^{1} (1 - x^2) \phi_i(x) \phi_j(x) \, \mathrm{d}x.$$

- (a) Show that  $\phi_k$  is an odd (even) function of x if k is odd (even) respectively.
- (b) Compute  $\phi_1(x), \phi_2(x), \phi_3(x)$ .
- (c) Hence derive the nodes and weights for a 3-point Gaussian quadrature formula  $\mathcal{G}_{2,w}(f)$  for integrals of the form

$$\int_{-1}^{1} (1 - x^2) f(x) \, \mathrm{d}x.$$

(d) For which of the following functions (if any) will the formula  $\mathcal{G}_{2,w}(f)$  give the exact integral? Justify your answer.

(i) 
$$f(x) = x^5 + x^4 + x^3$$
, (ii)  $f(x) = x^6 + x^5 + x^4$ , (iii)  $f(x) = x^7 + x^5 + x^4$ .