Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH2071-WE01

Title:

Mathematical Physics II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. A pointlike bead of mass m moves under gravity on a smooth wire bent into the shape of a helix described parametrically by $(x, y, z) = (\cos\theta, \sin\theta, \theta)$ with the z-axis pointing vertically upwards.

   (a) Find the kinetic energy of the particle in terms of θ?

   (b) Find the Lagrangian of the particle taking θ as the generalised co-ordinate.

   (c) Find the Euler-Lagrange equation for the particle.

   (d) Obtain the general solution to the Euler-Lagrange equation.

   (e) How long does the particle takes to travel from rest at $\theta = 2\pi$ to $\theta = 0$.

2. A system is described by a Lagrangian $L = L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n)$ which does not explicitly depend on time.

   (a) Use the Euler-Lagrange equations to show that

   $$E \equiv \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L$$

   is conserved.

   (b) Find E for the Lagrangian

   $$L_1 = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + (\dot{x} - \dot{y})\, z\,.$$

   (c) Which co-ordinates are ignorable for the Lagrangian L_1 and what are the corresponding conserved quantities? Use these to show that $\dot{x} + \dot{y}$ is conserved.

   (d) Find the Euler-Lagrange equations for L_1.

   (e) Compare the Euler-Lagrange equations for L_1 with those for

   $$L_2 = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \dot{z}(x - y)\,.$$

3. The Lagrangian of a system with generalised co-ordinates x and y is

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 - 5x^2 - 4xy - 2y^2\right)$$

(a) Find the Euler-Lagrange equations for x and y.

(b) Write these Euler-Lagrange equations in the matrix form

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} = -\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}$$

and find the two by two matrix $\mathbf{A}$.

(c) Find the eigenvalues and the eigenvectors of the matrix $\mathbf{A}$.

(d) Use these to construct the general solution of the Euler-Lagrange equation for the real column vector

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

(e) Initially the values of x and y and its derivative are $(x, y) = (2, 1)$ and $(\dot{x}, \dot{y}) = (0, 0)$. Find their subsequent values.

4. The orthonormal Hamiltonian eigenfunctions of an infinite potential well with $V(x) = 0$ in the region $0 \le x \le L$ are

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \qquad n \in \mathbb{Z}_{>0}\,.$$

At a fixed time, the wavefunction is $\psi(x) = C(\phi_1(x) + e^{i\delta}\phi_2(x))$ where $\delta \in \mathbb{R}$.

(a) Fix the normalization C.

(b) Calculate the energy expectation value $\langle H\rangle$.

(c) Calculate the position expectation value $\langle x\rangle$.

(d) Which of these measurements can detect the phase $\mathrm{e}^{i\delta}$?

You may use the definite integral $\displaystyle\int_0^\pi dy\, y\cos(ny) = \frac{(-1)^n - 1}{n^2}$ *for* $n \in \mathbb{Z}_{>0}$.

5. (a) What is Schrödinger's equation for a hamiltonian H?

   (b) Prove that the expectation value of a hermitian operator $\mathcal{O}$ obeys

$$\partial_t \langle \mathcal{O} \rangle = \frac{i}{\hbar} \langle [H, \mathcal{O}] \rangle \,.$$

   (c) For a particle of mass m in a potential $V(x)$, show that

$$\begin{aligned} \partial_t \langle x \rangle &= p/m \\ \partial_t \langle p \rangle &= -\langle \partial_x V(x) \rangle \,. \end{aligned}$$

   (d) What is the physical significance of the results in parts (b) and (c)?

6. Consider the wavefunction

$$\psi(x) = C e^{-\lambda |x| / \hbar} \,.$$

   (a) Determine C and sketch the probability density $|\psi(x)|^2$.

   (b) Compute the momentum space wavefunction $\widetilde{\psi}(p)$ and sketch the momentum probability density $|\widetilde{\psi}(p)|^2$.

   (c) What is the most likely value of momentum?

The momentum space wavefunction is defined by $\widetilde{\psi}(p) = \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int_{-\infty}^{\infty} e^{-ipx/\hbar} \phi(x)$.

SECTION B

7. A particular Lagrangian is unchanged under the infinitesimal transformation of generalised co-ordinates

$$q_i \to q_i' = q_i + \epsilon a_i(q_1, .., q_n), \quad \dot{q}_i \to \dot{q}_i' = \dot{q}_i + \epsilon \dot{a}_i(q_1, .., q_n)\,,$$

i.e.

$$L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n) \to L(q_1', .., q_n', \dot{q}_1', .., \dot{q}_n') = L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n)$$

up to and including terms linear in the infinitesimal parameter ϵ.

(a) Let δL denote the difference

$$\delta L = L(q_1', .., q_n', \dot{q}_1', .., \dot{q}_n') - L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n) = 0\,.$$

Using the chain-rule obtain another expression for δL in terms of a_i, $\dot{a}_i$ and partial derivatives of L. Hence obtain an equation involving a_i, $\dot{a}_i$ and the partial derivatives of L. Use this and the Euler-Lagrange equations to show that $\dot{Q} = 0$ where

$$Q = \sum_{i=1}^{n} a_i \frac{\partial L}{\partial \dot{q}_i}\,.$$

(b) In the Hamiltonian formulation the generalised velocities $\dot{q}_i$ are replaced by generalised momenta p_i. What is the definition of the generalised momenta? Use this definition to express Q in terms of generalised co-ordinates and momenta.

(c) Define the Poisson bracket $\{A, B\}$ of two functions of co-ordinates and momenta A and B and use Hamilton's equations of motion to show that provided A has no explicit dependence on time

$$\dot{A} = \{A, H\}$$

where H is the Hamiltonian.

(d) B is called the generator of an infinitesimal transformation if any function of the co-ordinates and momenta A transforms as

$$A \to A' = A + \epsilon\{A, B\}\,.$$

Show that under the infinitesimal transformation generated by Q the co-ordinates transform as

$$q_i \to q_i' = q_i + \epsilon a_i(q_1, .., q_n)\,.$$

How do the p_i transform? Show that H is unchanged under this transformation.

(e) A particle moves in three dimensions with Cartesian co-ordinates x_i. How do these co-ordinates transform under the infinitesimal transformation generated by $b_i J_i$ where

$$J_i = \epsilon_{ijk} x_j p_k\,,$$

b_i are the components of a constant vector and the summation convention has been used?

8. The displacement $u(x,t)$ of a particular string satisfies the wave-equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

and the energy of the portion of the string between $x = a$ and $x = b$ is given by

$$E(a,b) = \frac{1}{2}\int_a^b \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right) dx.$$

(a) Use the wave-equation to show that

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)$$

subject to the second partial derivatives of u being continuous.

(b) Hence show that

$$\frac{d}{dt}E(a,b) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_a^b .$$

(c) Define r and s by

$$r = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}, \quad s = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} .$$

Show that for any differentiable function H

$$\frac{\partial H(r)}{\partial t} = \frac{\partial H(r)}{\partial x}, \quad \frac{\partial H(s)}{\partial t} = -\frac{\partial H(s)}{\partial x} ,$$

and also show that this implies

$$\frac{d}{dt}\int_a^b H(r)\,dx = [H(r)]_{x=a}^{x=b}, \quad \frac{d}{dt}\int_a^b H(s)\,dx = -[H(s)]_{x=a}^{x=b} .$$

(d) The string is physically altered so that the wave equation is modified to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sin u$$

and as a consequence the energy is modified to

$$E(a,b) = \frac{1}{2}\int_a^b \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + V(u) \right) dx.$$

for some function $V(u)$. Find this function given that the energy still satisfies

$$\frac{d}{dt}E(a,b) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_a^b .$$

(e) Solutions to the modified wave equation can be found in the form $u = f(x-ct)$ provided f satisfies a second order ordinary differential equation. Find this equation and show that it implies that

$$\frac{1}{2}f'(x)^2 + \frac{\cos f(x)}{c^2-1}$$

is independent of x.

9. Consider a particle of mass m in a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. The wavefunction $\psi(x,t)$ is a normalized eigenfunction of the annihilation operator

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip)$$

with eigenvalue $\alpha(t) = \alpha_0 e^{-i\omega t}$ where α_0 is a real constant.

(a) Verify the commutation relation $[a, a^\dagger] = 1$.

(b) Express the hamiltonian $H = p^2/(2m) + V(x)$ in terms of a, $a^\dagger$ and hence show that

$$\langle H \rangle = \hbar\omega(\alpha_0^2 + \frac{1}{2}) .$$

(c) Express position and momentum in terms of a,$a^\dagger$ and hence show that

$$\langle x \rangle = \sqrt{\frac{2\hbar}{m\omega}}\,\alpha_0 \cos(\omega t) \qquad \langle p \rangle = -\sqrt{2m\hbar\omega}\,\alpha_0 \sin(\omega t) .$$

Verify that these expectation values solve the classical equations of motion.

(d) Compute the uncertainties Δx and Δp and show that they saturate Heisenberg's uncertaintly principle $\Delta x \Delta p \geq \hbar/2$ at all times t.

(e) Discuss the physical interpretation of this wavefunction.

10. Consider the finite barrier potential

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 \leq x \leq L \\ 0 & x > L \end{cases}$$

and the following ansatz for the wavefunction,

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & x < 0 \\ A + Bx & 0 \leq x \leq L \\ te^{ikx} & x > L \end{cases} ,$$

where $k = \sqrt{2mV_0}/\hbar$ and r, A, B, and t are constants to be determined.

(a) Show that this is an eigenfunction of the hamiltonian with energy $E = V_0$.

(b) What boundary conditions should the wavefunction obey at $x = 0$ and $x = L$?

(c) Determine the coefficients r, t, A, B and show that $|r|^2 + |t|^2 = 1$.

(d) What is the physical significance of the quantities $|r|^2$, $|t|^2$?

(e) Sketch $|r|^2$, $|t|^2$ as a function of the barrier length L and discuss their behaviour as $L \ll 1/k$ and $L \gg 1/k$.