

EXAMINATION PAPER

Examination Session: May

2018

Year:

Exam Code:

MATH2071-WE01

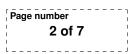
Title:

Mathematical Physics II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best FOUR answers from Section A and the best THREE answers from Section B. Questions in Section B carry TWICE as many marks as those in Section A.
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Revision:



SECTION A

- 1. A pointlike bead of mass m moves under gravity on a smooth wire bent into the shape of a helix described parametrically by $(x, y, z) = (\cos \theta, \sin \theta, \theta)$ with the z-axis pointing vertically upwards.
 - (a) Find the kinetic energy of the particle in terms of θ ?
 - (b) Find the Lagrangian of the particle taking θ as the generalised co-ordinate.
 - (c) Find the Euler-Lagrange equation for the particle.
 - (d) Obtain the general solution to the Euler-Lagrange equation.
 - (e) How long does the particle takes to travel from rest at $\theta = 2\pi$ to $\theta = 0$.
- 2. A system is described by a Lagrangian $L = L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n)$ which does not explicitly depend on time.
 - (a) Use the Euler-Lagrange equations to show that

$$E \equiv \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L$$

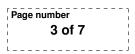
is conserved.

(b) Find E for the Lagrangian

$$L_1 = \frac{m}{2} \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + \left(\dot{x} - \dot{y} \right) z \,.$$

- (c) Which co-ordinates are ignorable for the Lagrangian L_1 and what are the corresponding conserved quantities? Use these to show that $\dot{x} + \dot{y}$ is conserved.
- (d) Find the Euler-Lagrange equations for L_1 .
- (e) Compare the Euler-Lagrange equations for L_1 with those for

$$L_2 = \frac{m}{2} \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - \dot{z}(x - y) \,.$$





3. The Lagrangian of a system with generalised co-ordinates x and y is

$$L = \frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 - 5x^2 - 4xy - 2y^2 \right)$$

- (a) Find the Euler-Lagrange equations for x and y.
- (b) Write these Euler-Lagrange equations in the matrix form

$$\frac{d^2}{dt^2} \left(\begin{array}{c} x\\ y \end{array}\right) = -\mathbf{A} \left(\begin{array}{c} x\\ y \end{array}\right)$$

and find the two by two matrix \mathbf{A} .

- (c) Find the eigenvalues and the eigenvectors of the matrix **A**.
- (d) Use these to construct the general solution of the Euler-Lagrange equation for the real column vector

$$\left(\begin{array}{c} x\\ y\end{array}\right)$$

(e) Initially the values of x and y and its derivative are (x, y) = (2, 1) and $(\dot{x}, \dot{y}) = (0, 0)$. Find their subsequent values.

4. The orthonormal Hamiltonian eigenfunctions of an infinite potential well with V(x) = 0 in the region $0 \le x \le L$ are

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad n \in \mathbb{Z}_{>0}.$$

At a fixed time, the wavefunction is $\psi(x) = C(\phi_1(x) + e^{i\delta}\phi_2(x))$ where $\delta \in \mathbb{R}$.

- (a) Fix the normalization C.
- (b) Calculate the energy expectation value $\langle H \rangle$.
- (c) Calculate the position expectation value $\langle x \rangle$.
- (d) Which of these measurements can detect the phase $e^{i\delta}$?

You may use the definite integral
$$\int_0^{\pi} dy \, y \cos(ny) = \frac{(-1)^n - 1}{n^2}$$
 for $n \in \mathbb{Z}_{>0}$.

- 5. (a) What is Schrödinger's equation for a hamiltonian H?
 - (b) Prove that the expectation value of a hermitian operator \mathcal{O} obeys

$$\partial_t \langle \mathcal{O} \rangle = \frac{i}{\hbar} \langle [H, \mathcal{O}] \rangle \,.$$

(c) For a particle of mass m in a potential V(x), show that

$$\begin{split} \partial_t \langle x \rangle &= p/m \\ \partial_t \langle p \rangle &= - \langle \partial_x V(x) \rangle \,. \end{split}$$

- (d) What is the physical significance of the results in parts (b) and (c)?
- 6. Consider the wavefunction

$$\psi(x) = C e^{-\lambda|x|/\hbar}$$

- (a) Determine C and sketch the probability density $|\psi(x)|^2$.
- (b) Compute the momentum space wavefunction $\tilde{\psi}(p)$ and sketch the momentum probability density $|\tilde{\psi}(p)|^2$.
- (c) What is the most likely value of momentum?

The momentum space wavefunction is defined by $\widetilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \phi(x).$

SECTION B

7. A particular Lagrangian is unchanged under the infinitesimal transformation of generalised co-ordinates

$$q_i \to q'_i = q_i + \epsilon a_i(q_1, ..., q_n), \quad \dot{q}_i \to \dot{q}'_i = \dot{q}_i + \epsilon \dot{a}_i(q_1, ..., q_n),$$

i.e.

$$L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n) \to L(q'_1, .., q'_n, \dot{q}'_1, .., \dot{q}'_n) = L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n)$$

up to and including terms linear in the infinitesimal parameter ϵ .

(a) Let δL denote the difference

$$\delta L = L(q'_1, ..., q'_n, \dot{q}'_1, ..., \dot{q}'_n) - L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n) = 0.$$

Using the chain-rule obtain another expression for δL in terms of a_i , \dot{a}_i and partial derivatives of L. Hence obtain an equation involving a_i , \dot{a}_i and the partial derivatives of L. Use this and the Euler-Lagrange equations to show that $\dot{Q} = 0$ where

$$Q = \sum_{i=1}^{n} a_i \frac{\partial L}{\partial \dot{q}_i}.$$

- (b) In the Hamiltonian formulation the generalised velocities \dot{q}_i are replaced by generalised momenta p_i . What is the definition of the generalised momenta? Use this definition to express Q in terms of generalised co-ordinates and momenta.
- (c) Define the Poisson bracket $\{A, B\}$ of two functions of co-ordinates and momenta A and B and use Hamilton's equations of motion to show that provided A has no explicit dependence on time

$$\dot{A} = \{A, H\}$$

where H is the Hamiltonian.

(d) B is called the generator of an infinitesimal transformation if any function of the co-ordinates and momenta A transforms as

$$A \to A' = A + \epsilon \{A, B\}$$

Show that under the infinitesimal transformation generated by Q the co-ordinates transform as

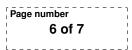
$$q_i \rightarrow q'_i = q_i + \epsilon a_i(q_1, ..., q_n)$$
.

How do the p_i transform? Show that H is unchanged under this transformation.

(e) A particle moves in three dimensions with Cartesian co-ordinates x_i . How do these co-ordinates transform under the infinitesimal transformation generated by $b_i J_i$ where

$$J_i = \epsilon_{ijk} x_j p_k \,,$$

 b_i are the components of a constant vector and the summation convention has been used?



- Exam code MATH2071-WE01
- 8. The displacement u(x,t) of a particular string satisfies the wave-equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

and the energy of the portion of the string between x = a and x = b is given by

$$E(a,b) = \frac{1}{2} \int_{a}^{b} \left(\left(\frac{\partial u}{\partial t} \right)^{2} + \left(\frac{\partial u}{\partial x} \right)^{2} \right) dx.$$

(a) Use the wave-equation to show that

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)$$

subject to the second partial derivatives of u being continuous.

(b) Hence show that

$$\frac{d}{dt}E(a,b) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_{a}^{b}$$

(c) Define r and s by

$$r = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}, \quad s = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}.$$

Show that for any differentiable function H

$$\frac{\partial H(r)}{\partial t} = \frac{\partial H(r)}{\partial x}, \quad \frac{\partial H(s)}{\partial t} = -\frac{\partial H(s)}{\partial x},$$

and also show that this implies

$$\frac{d}{dt} \int_{a}^{b} H(r) \, dx = [H(r)]_{x=a}^{x=b}, \quad \frac{d}{dt} \int_{a}^{b} H(s) \, dx = -[H(s)]_{x=a}^{x=b}.$$

(d) The string is physically altered so that the wave equation is modified to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sin u$$

and as a consequence the energy is modified to

$$E(a,b) = \frac{1}{2} \int_{a}^{b} \left(\left(\frac{\partial u}{\partial t} \right)^{2} + \left(\frac{\partial u}{\partial x} \right)^{2} + V(u) \right) dx.$$

for some function V(u). Find this function given that the energy still satisfies

$$\frac{d}{dt}E(a,b) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_{a}^{b}$$

(e) Solutions to the modified wave equation can be found in the form u = f(x - ct) provided f satisfies a second order ordinary differential equation. Find this equation and show that it implies that

$$\frac{1}{2}f'(x)^2 + \frac{\cos f(x)}{c^2 - 1}$$

is independent of x.

- Exam code MATH2071-WE01
- 9. Consider a particle of mass m in a harmonic oscilator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. The wavefunction $\psi(x, t)$ is a normalized eigenfunction of the annihilation operator

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip)$$

with eigenvalue $\alpha(t) = \alpha_0 e^{-i\omega t}$ where α_0 is a real constant.

- (a) Verify the commutation relation $[a, a^{\dagger}] = 1$.
- (b) Express the hamiltonian $H = p^2/(2m) + V(x)$ in terms of a, a^{\dagger} and hence show that

$$\langle H \rangle = \hbar \omega (\alpha_0^2 + \frac{1}{2}).$$

(c) Express position and momentum in terms of $a_{a}a^{\dagger}$ and hence show that

$$\langle x \rangle = \sqrt{\frac{2\hbar}{m\omega}} \alpha_0 \cos(\omega t) \qquad \langle p \rangle = -\sqrt{2m\hbar\omega} \alpha_0 \sin(\omega t) \,.$$

Verify that these expectation values solve the classical equations of motion.

- (d) Compute the uncertainties Δx and Δp and show that they saturate Heisenberg's uncertainty principle $\Delta x \Delta p \ge \hbar/2$ at all times t.
- (e) Discuss the physical interpretation of this wavefunction.
- 10. Consider the finite barrier potential

$$V(x) = \begin{cases} 0 & x < 0\\ V_0 & 0 \le x \le L\\ 0 & x > L \end{cases}$$

and the following ansatz for the wavefunction,

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & x < 0\\ A + Bx & 0 \le x \le L\\ te^{ikx} & x > L \end{cases},$$

where $k = \sqrt{2mV_0}/\hbar$ and r, A, B, and t are constants to be determined.

- (a) Show that this is an eigenfunction of the hamiltonian with energy $E = V_0$.
- (b) What boundary conditions should the wavefunction obey at x = 0 and x = L?
- (c) Determine the coefficients r, t, A, B and show that $|r|^2 + |t|^2 = 1$.
- (d) What is the physical significance of the quantities $|r|^2$, $|t|^2$?
- (e) Sketch $|r|^2$, $|t|^2$ as a function of the barrier length L and discuss their behaviour as $L \ll 1/k$ and $L \gg 1/k$.