Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH2581-WE01

Title:

Algebra II

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. Find the monic greatest common divisor $d(x)$ in $(\mathbb{Z}/3)[x]$ of the polynomials

$$f(x) = x^5 + \overline{2}x^3 + x^2 + x + \overline{1}, \qquad g(x) = x^4 + \overline{2}x^3 + x + \overline{1}$$

and find polynomials $A(x), B(x) \in (\mathbb{Z}/3)[x]$ such that

$$A(x)f(x) + B(x)g(x) = d(x).$$

2. (a) Let R be a commutative ring and let I be an ideal in R. Let $a, b \in R$ be elements in R.

   i. State the definition of the ideal generated by a and b, denoted by (a, b).

   ii. Prove that if a and b are elements of R such that $a, b \in I$, then

$$(a, b) \subseteq I.$$

   (b) Let $R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$. Then prove the following equality of ideals

$$(3 + \sqrt{5}) = (13 + 7\sqrt{5}, 8 + 4\sqrt{5}).$$

3. Let R be the quotient ring $\mathbb{Z}/2[x]/(x^3+x+\overline{1})$. Let $g(x) = x^5+x^4+x+\overline{1} \in (\mathbb{Z}/2)[x]$.

   (a) Without proof, list all the elements of R.

   (b) Find a polynomial $h(x) \in (\mathbb{Z}/2)[x]$ of degree ≤ 2 such that $\overline{g(x)} = \overline{h(x)}$ in R.

   (c) Find a multiplicative inverse of $\overline{g(x)}$ in R.

4. (a) Let $\phi\colon G \to H$ be a group homomorphism. Show that $\ker\phi$ is a normal subgroup of G.
   (b) State the First Isomorphism Theorem for groups and Lagrange's theorem.
   (c) Let G be a finite group of order 21 and let K be a finite group of order 49. Suppose that G does not have a normal subgroup of order 3. Then show that the only group homomorphism from G to K is the trivial map: $\phi(g) = e_K$ for all $g \in G$.

5. For each of the following pairs of groups, either write down an isomorphism (you do not have to prove that it is an isomorphism) between the two groups, or prove that the two groups are not isomorphic by identifying an invariant quantity.
   (a) S_3 and D_3.
   (b) D_3 and $\mathbb{Z}/3$.
   (c) $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$.
   (d) $\mathbb{Z}/2 \times \mathbb{Z}/3$ and $\mathbb{Z}/6$.
   (e) Q_8 and D_4.

   Here S_3 is the symmetric group on 3 letters, D_n is the dihedral group of symmetries of a regular n-gon, and $Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle$, with 1 the identity element, is the quaternion group of order 8.

6. Let G be an abelian group and let $f\colon G \to \mathbb{Z}$ be a surjective group homomorphism. Prove that we have an isomorphism of groups $G \cong \ker(f) \times \mathbb{Z}$. To start, choose an element $s \in G$ such that $f(s) = 1 \in \mathbb{Z}$, and use s to write down a homomorphism from G to $\ker(f) \times \mathbb{Z}$, or vice versa, and then show that the map you write down is an isomorphism.

SECTION B

7. Suppose R is an integral domain, and let a and b be irreducible elements of R. The elements a and b are said to be *associates* if there exists a unit $u \in R$ satisfying $a = ub$.

   (a) Show that if a and b are both divisible by $p \in R$, then either a and b are associates or p is a unit.

   (b) Suppose I is a principal ideal of R containing a and b. Show that if a and b are not associates, then $I = R$.

   (c) Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$. Let $N(a + b\sqrt{-5}) = a^2 + 5b^2$ be a norm on R. Show that $N(u) = 1$ for any unit $u \in R$. You may use here that the norm is multiplicative, i.e., $N(xy) = N(x)N(y)$ for every $x, y \in R$.

   (d) With the help of part (c), show that the elements 2 and $1+\sqrt{-5}$ of the integral domain $\mathbb{Z}[\sqrt{-5}]$ are not associates.

   (e) Using the previous parts, show that the ideal $(2, 1 + \sqrt{-5})$ of $\mathbb{Z}[\sqrt{-5}]$ is not principal. You may assume that 2 and $1 + \sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$.

8. Let $f(x) = x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$.

   (a) Prove that the polynomial $\bar{f}(x) = x^4+x^3+x^2+x+\bar{1} \in (\mathbb{Z}/2)[x]$ is an irreducible polynomial.

   (b) Using part (a) or otherwise, prove that the polynomial $f(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$.

   (c) Let $\alpha \in \mathbb{C}$ be a root of $f(x)$. Let

   $$\mathbb{Q}[\alpha] = \{a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 : a_0, a_1, a_2, a_3 \in \mathbb{Q}\}$$

   be a subset of $\mathbb{C}$. Prove that $\mathbb{Q}[\alpha]$ is a subring of $\mathbb{C}$.

   (d) Using the evaluation map $\phi : \mathbb{Q}[x] \to \mathbb{C}$, defined by $\phi(g(x)) = g(\alpha)$, prove that $\mathbb{Q}[\alpha] \cong \mathbb{Q}[x]/(f(x))$. Prove that $\mathbb{Q}[\alpha]$ a field. You may use that ϕ is a homomorphism.

   (e) Prove that $\mathbb{Q}[x]/(x^5 - 1) \cong \mathbb{Q} \times \mathbb{Q}[\alpha]$.

9. (a) Compute the conjugacy classes in D_{10}, the dihedral group with 20 elements.

   (b) Find a non-cyclic subgroup of D_{10} with 10 elements.

   (c) Find a normal subgroup of D_{10} with cardinality 5. Construct a surjective homomorphism from D_{10} to a group of cardinality 4.

   (d) Show that there is no normal subgroup of D_{10} with cardinality 4.

10. (a) State the Burnside lemma for counting the number of orbits of an action of a finite group G on a set X. Assuming the orbit-stabiliser theorem, give the proof of the Burnside lemma.

    (b) How many essentially different ways are there to colour the faces of a cube in some combination of red and yellow? Here we say that two colourings are the same if one can be rotated to the other. [Hint: to apply the Burnside lemma, let X be the set of possible colourings of the cube in one position, and let G be the order 24 group of rotational symmetries of the cube.]