Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH2617-WE01

Title: Elementary Number Theory II

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **TWO** answers from Section A and the best **TWO** answers from Section B. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. (a) Can 15400 be written as the sum of two squares? Justify your answer.

   (b) Write 2952 as the sum of two squares.

2. (a) Prove that if (x, y, z) is a primitive Pythagorean triple, then x and y can't both be even.

   (b) Prove that if (x, y, z) is a Pythagorean triple, then x and y can't both be odd.

   (c) Assume that $(154, y, z)$ is a primitive Pythagorean triple. Find y and z. (Hint: Use the explicit formula for primitive Pythagorean triples.)

3. (a) Find the orders of all integers between 1 and 10 modulo 11, that is, find $\text{ord}_{11}(n)$, for $n = 1, 2, \ldots, 10$.

   (b) Which integers between 1 and 10 are primitive roots modulo 11?

   (c) How many primitive roots are there modulo 41? Justify. (Hint: This is part of the statement and proof of the existence of primitive roots in the lectures.)

SECTION B

4. Let φ denote the Euler φ-function.

   (a) State Euler's theorem.

   (b) Compute $\varphi(51000)$.

   (c) Find a solution $x \in \mathbb{N}$ to $x^3 \equiv 2 \pmod{51}$.

   (d) Find the two last digits of 33^{4444}.

5. (a) You are given that 104281 is a prime number. Evaluate the Legendre symbol $\left(\frac{-1}{104281}\right)$ using Euler's criterion, or otherwise.

   (b) Evaluate the Legendre symbols $\left(\frac{2}{3}\right)$ and $\left(\frac{2}{23}\right)$.

   (c) Determine whether the congruence $x^2 \equiv -2 \pmod{71}$ has a solution or not.

   (d) Suppose that $p > 3$ is a prime. Show that if $p \equiv \pm 1 \pmod{12}$, then $\left(\frac{3}{p}\right) = 1$.

6. (a) Find the simple continued fraction expansion of $\sqrt{41}$ and write it in the form $[a_0; \overline{a_1, \ldots, a_n}]$.

   (b) Compute the numbers p_k and q_k associated to the simple continued fraction of $\sqrt{41}$, for $k = 0, 1, 2, 3, 4$, and find a rational number $a \in \mathbb{Q}$ such that

   $$\left|\sqrt{41} - a\right| < 0.0002.$$

   (c) Let $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for all $n \in \mathbb{N}$ with $n \geq 2$. Then $F_0, F_1, F_2, \ldots$ is the so-called Fibonacci sequence. Find the simple continued fraction $[a_0; \overline{a_1, \ldots, a_n}]$ whose convergent p_k/q_k equals F_{k+1}/F_k, for all $k = 0, 1, 2, \ldots$.

   (d) Find $\lim_{k\to\infty} F_k/F_{k-1}$.