

EXAMINATION PAPER

Examination Session: May

2018

Year:

Exam Code:

MATH2647-WE01

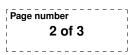
Title:

Probability II

Time Allowed:	2 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for the best TWO and the best TWO answers from Sec Questions in Section B carry ONE an marks as those in Section A.	tion B.	
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Revision:



SECTION A

- 1. (a) State the Borel-Cantelli lemmas. In your answer you should carefully define any notation you use.
 - (b) In a probabilistic experiment, biased coins are flipped sequentially and independently. Assume that the n^{th} coin shows heads with probability p_n and tails with probability $1 p_n$. Let $A_n = \{n^{\text{th}} \text{ coin flip is head}\}$. In each of the following cases, find $\mathbb{P}[\limsup A_n]$:

(i)
$$p_n = \frac{1}{2}$$
,
(ii) $p_n = \frac{1}{n}$,
(iii) $p_n = \frac{1}{n^2}$,
(iv) $p_n = \frac{1}{n \log n}$.

- 2. (a) Give a careful definition of a recurrent state and a transient state for a Markov chain $(X_n)_{n\geq 0}$ on a state space I.
 - (b) Consider a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$\begin{pmatrix} 2/3 & 1/3 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix}.$$

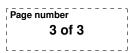
- (i) Find all communicating classes. Explain which classes are closed and which are not.
- (ii) Determine the recurrent and transient states for this Markov chain.
- 3. (a) State the monotone convergence theorem and the dominated convergence theorem.
 - (b) Let $\Omega = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$ with, for $\omega \in \Omega$,

$$\mathbb{P}[\{\omega\}] = \begin{cases} 3^{-\omega} & \text{if } \omega \neq 0\\ \frac{1}{2} & \text{if } \omega = 0 \end{cases}$$

and let, for $n \ge 1$,

$$X_n(\omega) = \begin{cases} 3^n & \text{if } \omega = n \\ 0 & \text{otherwise.} \end{cases}$$

- (i) Show that $X_n \to 0$ almost surely.
- (ii) Compute $\mathbb{E}[X_n]$.
- (iii) Compute $\lim_{n\to\infty} \mathbb{E}[X_n]$ and $\mathbb{E}[\lim_{n\to\infty} X_n]$. Can you apply the monotone convergence theorem to the sequence $(X_n)_{n\geq 1}$? Can you apply the dominated convergence theorem to the sequence $(X_n)_{n\geq 1}$? In each case, explain why or why not.



SECTION B

- 4. Let $(X_n)_{n\geq 1}$ be independent and identically distributed random variables with values in $\{0, 1, 2, ...\}$ and let $N \geq 0$ be an integer valued random variable independent of $(X_n)_{n\geq 1}$. Let $G_X(s)$ be the generating function for X_1 and $G_N(s)$ be the generating function for N.
 - (a) Prove that $S_N = X_1 + \cdots + X_N$ has generating function $G_{S_N}(s) = G_N(G_X(s))$.
 - (b) Using generating functions, show that

$$\mathbb{E}[S_N] = \mathbb{E}[N]\mathbb{E}[X]$$

and

$$\operatorname{Var}[S_N] = \mathbb{E}[N]\operatorname{Var}[X] + \operatorname{Var}[N]\mathbb{E}[X]^2.$$

- 5. Let $(X_n)_{n>0}$ and $(Y_n)_{n>0}$ be sequences of random variables.
 - (a) Carefully define convergence in probability, convergence almost surely, and convergence in L^r for r > 0.
 - (b) Show that if $X_n \to X$ almost surely as $n \to \infty$ and $Y_n \to Y$ almost surely as $n \to \infty$, then $X_n Y_n \to XY$ almost surely as $n \to \infty$.
 - (c) If $X_n \to X$ in L^r as $n \to \infty$ and $Y_n \to Y$ in L^r as $n \to \infty$, does $X_n Y_n \to XY$ in L^r as $n \to \infty$? Justify your answer by giving a proof or a counterexample.
 - (d) Show that $X_n \to X$ almost surely as $n \to \infty$ whenever $\sum_{n=1}^{\infty} \mathbb{E}[|X_n X|^2] < \infty$.
- 6. (a) Let $(X_n)_{n\geq 0}$ be a Markov chain on a state space I with transition matrix $P = (p_{ij})$. Let $p_{ij}^{(n)}$ denote the *n*-step transition probability from *i* to *j*. Prove that

$$p_{ij}^{(m+n)} = \sum_{k \in I} p_{ik}^{(n)} p_{kj}^{(m)} \quad \text{for all } m, n \ge 1.$$

- (b) A gambler has $\pounds 2$ and needs to increase his money to $\pounds 10$ in a hurry. He decides to play a series of independent rounds at a casino with the following bets: the player bets a stake of $\pounds x$, if he wins, which happens with probability 1/4, he receives his stake back and a further $\pounds x$. If he loses, which happens with probability 1/2, he receives only his stake back. If he ties, which happens with probability 1/2, he receives only his stake back. The gambler decides to use a bold strategy in which he bets all his money if he has $\pounds 4$ or less and otherwise he bets just enough to increase his capital, if he wins, to $\pounds 10$. Find the probability that the gambler will achieve his aim.
- (c) Show that every Markov chain on a finite state space has at least one closed communicating class. Find an example of a Markov chain which has no closed communicating classes.