Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH2647-WE01

Title:
Probability II

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **TWO** answers from Section A and the best **TWO** answers from Section B. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. (a) State the Borel-Cantelli lemmas. In your answer you should carefully define any notation you use.

   (b) In a probabilistic experiment, biased coins are flipped sequentially and independently. Assume that the n^{th} coin shows heads with probability p_n and tails with probability $1 - p_n$. Let $A_n = \{n^{\text{th}} \text{ coin flip is head}\}$. In each of the following cases, find $\mathbb{P}[\limsup A_n]$:

   (i) $p_n = \frac{1}{2}$,
   (ii) $p_n = \frac{1}{n}$,
   (iii) $p_n = \frac{1}{n^2}$,
   (iv) $p_n = \frac{1}{n \log n}$.

2. (a) Give a careful definition of a recurrent state and a transient state for a Markov chain $(X_n)_{n \geq 0}$ on a state space I.

   (b) Consider a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$\begin{pmatrix} 2/3 & 1/3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

   (i) Find all communicating classes. Explain which classes are closed and which are not.
   (ii) Determine the recurrent and transient states for this Markov chain.

3. (a) State the monotone convergence theorem and the dominated convergence theorem.

   (b) Let $\Omega = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$ with, for $\omega \in \Omega$,

$$\mathbb{P}[\{\omega\}] = \begin{cases} 3^{-\omega} & \text{if } \omega \neq 0 \\ \frac{1}{2} & \text{if } \omega = 0 \end{cases}$$

   and let, for $n \geq 1$,

$$X_n(\omega) = \begin{cases} 3^n & \text{if } \omega = n \\ 0 & \text{otherwise.} \end{cases}$$

   (i) Show that $X_n \to 0$ almost surely.
   (ii) Compute $\mathbb{E}[X_n]$.
   (iii) Compute $\lim_{n\to\infty} \mathbb{E}[X_n]$ and $\mathbb{E}[\lim_{n\to\infty} X_n]$. Can you apply the monotone convergence theorem to the sequence $(X_n)_{n\geq 1}$? Can you apply the dominated convergence theorem to the sequence $(X_n)_{n\geq 1}$? In each case, explain why or why not.

SECTION B

4. Let $(X_n)_{n\geq 1}$ be independent and identically distributed random variables with values in $\{0,1,2,\dots\}$ and let $N \geq 0$ be an integer valued random variable independent of $(X_n)_{n\geq 1}$. Let $G_X(s)$ be the generating function for X_1 and $G_N(s)$ be the generating function for N.

   (a) Prove that $S_N = X_1 + \cdots + X_N$ has generating function $G_{S_N}(s) = G_N(G_X(s))$.

   (b) Using generating functions, show that

   $$\mathbb{E}[S_N] = \mathbb{E}[N]\mathbb{E}[X]$$

   and

   $$\mathrm{Var}[S_N] = \mathbb{E}[N]\mathrm{Var}[X] + \mathrm{Var}[N]\mathbb{E}[X]^2.$$

5. Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be sequences of random variables.

   (a) Carefully define *convergence in probability*, *convergence almost surely*, and *convergence in L^r* for $r > 0$.

   (b) Show that if $X_n \to X$ almost surely as $n \to \infty$ and $Y_n \to Y$ almost surely as $n \to \infty$, then $X_n Y_n \to XY$ almost surely as $n \to \infty$.

   (c) If $X_n \to X$ in L^r as $n \to \infty$ and $Y_n \to Y$ in L^r as $n \to \infty$, does $X_n Y_n \to XY$ in L^r as $n \to \infty$? Justify your answer by giving a proof or a counterexample.

   (d) Show that $X_n \to X$ almost surely as $n \to \infty$ whenever $\sum_{n=1}^{\infty} \mathbb{E}[|X_n - X|^2] < \infty$.

6. (a) Let $(X_n)_{n\geq 0}$ be a Markov chain on a state space I with transition matrix $P = (p_{ij})$. Let $p_{ij}^{(n)}$ denote the n-step transition probability from i to j. Prove that

   $$p_{ij}^{(m+n)} = \sum_{k\in I} p_{ik}^{(n)} p_{kj}^{(m)} \quad \text{for all } m, n \geq 1.$$

   (b) A gambler has £2 and needs to increase his money to £10 in a hurry. He decides to play a series of independent rounds at a casino with the following bets: the player bets a stake of £x, if he wins, which happens with probability 1/4, he receives his stake back and a further £x. If he loses, which happens with probability 1/4, he loses his stake. If he ties, which happens with probability 1/2, he receives only his stake back. The gambler decides to use a bold strategy in which he bets all his money if he has £4 or less and otherwise he bets just enough to increase his capital, if he wins, to £10. Find the probability that the gambler will achieve his aim.

   (c) Show that every Markov chain on a finite state space has at least one closed communicating class. Find an example of a Markov chain which has no closed communicating classes.