Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH2657-WE01

Title: Special Relativity & Electromagnetism II

Time Allowed:	2 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for the best **TWO** answers from Section A and the best **TWO** answers from Section B. Questions in Section B carry **ONE and a HALF times** as many marks as those in Section A.

Revision:	

SECTION A

1. (a) If ρ is the density of electric charge and $\mathbf{J}$ is the current density the continuity equation states

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0\,.$$

Use this to derive a statement concerning the rate of change of charge in a fixed volume expressed as an integral and give its physical interpretation.

(b) State Ampère's law for magnetic fields in both integral and differential form and use the differential form to show that it is incompatible with the continuity equation when the fields and charge density vary in time. How did Maxwell remedy this?

2. (a) An electric charge of magnitude q is placed at the centre of the square base of a pyramid. The other faces of the pyramid are equilateral triangles. What is the flux of the electric field through one triangular face? Does this change when the equilateral triangles are replaced by isosceles triangles of arbitrary height?

(b) A wire carrying a current I in the direction of increasing y lies in the xy-plane along the line $x = 0$. Assuming that this produces a magnetic field which, at points on the plane is normal to the plane. find the magnitude and direction of this field at the point (x, y). A second wire, carrying current $2I$ in the same direction as the first, is placed in the plane along the line $x = 1$. Where in the xy-plane does the total magnetic field due to the two wires vanish?

3. Muons are unstable particles. Some muons are created in the atmosphere 800 metres above the surface of the Earth in a process that results in them having a kinetic energy in the rest-frame of the earth equal to two-thirds of their rest energy.

(a) What speed do the muons have relative to the earth? (Take $c = 3 \times 10^8\ m/s$.)

(b) If one of the muons, travelling vertically downwards, just reaches sea-level, how long is its lifetime in its own rest-frame? In this rest-frame, what length of atmosphere does it travel through?

SECTION B

4. (a) Show that the following two of Maxwell's equations

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}\,, \quad \nabla \cdot \mathbf{B} = 0\,,$$

are satisfied by taking

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\dot{\mathbf{A}} - \nabla\varphi\,.$$

and show that $\mathbf{E}$ and $\mathbf{B}$ are unchanged if $\mathbf{A}$ and φ are replaced by

$$\mathbf{A}' = \mathbf{A} + \nabla\psi, \quad \varphi' = \varphi - \dot{\psi}\,.$$

(b) Assuming that $\mathbf{A}$ and φ are related by

$$\nabla \cdot \mathbf{A} + \epsilon_0\mu_0\dot{\varphi} = 0$$

use the remaining Maxwell equations

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0\,, \quad \nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\dot{\mathbf{E}}\,,$$

to obtain a differential equation for φ and a differential equation for $\mathbf{A}$. What do these equations reduce to in free space where $\rho = 0$ and $\mathbf{J} = 0$?

(c) Assuming that (complex) solutions to these equations in free space can be found in the form

$$\mathbf{A} = \mathbf{V}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \quad \varphi = fe^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},$$

find the constants ω and f in terms of the constant vectors $\mathbf{k}$ and $\mathbf{V}$ and the constants of nature ϵ_0 and μ_0.

5. (a) Two events are observed to occur four seconds apart at the same place in an inertial frame. What is the value of the invariant interval between the events? Is it spacelike, timelike or null? What is the spatial separation of the events as measured in another inertial frame in which the time between them is measured to be five seconds? Take $c = 3 \times 10^8 \; m/s$.

   (b) Using the matrix
$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
write down the conditions on the four by four matrix $\mathbf{L}$ for $\mathbf{x} \to \mathbf{x}' = \mathbf{L}\mathbf{x}$ to be a proper orthochronous Lorentz transformation. Write these conditions in terms of the numbers a, b, c, and d if $\mathbf{L}$ takes the following form
$$\mathbf{L} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & d \end{pmatrix}$$
Show that these conditions are satisfied if $a = \cosh\theta = d$ and $b = \sinh\theta = c$.

   (c) A particle moves with speed u along the x^3-axis of the inertial frame with co-ordinates $\mathbf{x}$ so $x^3 = ux^0/c$. Use $\mathbf{L}$ to write this equation in the co-ordinates $\mathbf{x}'$ and hence obtain the speed of the particle in the frame with co-ordinates $\mathbf{x}'$ in terms of u and θ.

6. (a) What is the condition on the quantities $L^\mu{}_\nu$ for the transformation expressed in component form as
$$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu$$
to be a Lorentz transformation? A particle with rest-mass m moves along the parametrised curve $x^\mu = x^\mu(u)$ and the proper time is defined to be
$$\tau(u) = \frac{1}{c}\int_{u_0}^{u} \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}}\, du\,.$$
Show that the proper time is invariant under Lorentz transformations. (You may assume that u can be chosen to be invariant under Lorentz transformations).

   (b) How does a four-vector with components V^μ transform under a Lorentz transformation? Show that the four-velocity of the particle
$$\frac{dx^\mu}{d\tau}$$
transforms as a four-vector. Find the Lorentz invariant dot product of the four-velocity with itself (hint: differentiate the expression in part (a) for $\tau(u)$ with respect to u and then choose $u = \tau$) . How is the four-momentum of the particle related to its four-velocity?

   (c) Two particles of equal mass collide and fuse to form a single particle of three times the rest-mass of either particle. What were their speeds in the rest-frame of the final particle and what were their speeds relative to each other? (You can leave your answers in the form of a multiple of c).