Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH2667-WE01

Title:
Monte Carlo II

Time Allowed:	1 hour 30 minutes	
Additional Material provided:	Tables: Normal.	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **ONE** answer from Section B. Questions in Section B carry **THREE TIMES** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. In a Monte Carlo simulation, 500 values were sampled for a random variable Y and tabulated as follows:

Y	1	4	8
Count	30	90	380

   (a) Estimate the probabilities $\mathrm{P}[Y = 1]$ and $\mathrm{P}[Y \leq 4]$ and calculate a 95% confidence interval for each of them.

   (b) Suppose that we want to know $\phi = \mathrm{E}[\log_2 Y]$. Estimate ϕ and calculate a 99% confidence interval for ϕ.

   (c) Suppose that it is known that 2 is also a possible value for Y but just didn't appear in the sample. Estimate the probability that $Y = 2$.

   What goes wrong with the usual confidence interval calculation?

   Describe another way to say something useful about the possible size of the probability.

2. (a) Explain how to construct a random vector $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ having a bivariate normal distribution from a random vector $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ where Z_1 and Z_2 are independent $\mathrm{N}(0, 1)$, standard normal, random variables.

   (b) Show how the means and variances of X_1 and X_2, and the covariance between X_1 and X_2, depend on the elements of the construction.

   (c) Let the mean vector and variance-covariance matrix of $\boldsymbol{X}$ be respectively $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 16 & 12 \\ 12 & 11.25 \end{pmatrix}$. Suppose that -0.6 and 1.1 are two values sampled from the standard normal distribution. Use these to sample a value for $\boldsymbol{X}$.

3. (a) For a homogeneous Poisson process with rate λ, show that the distribution of the time, X_1, to the first event is $\mathrm{Exp}(\lambda)$. You may assume that the number of events, $N(t)$, up to time t has the Poisson distribution with mean λt.

   Explain informally why the time, X_2, between the first and second events is independent of X_1 and has the same distribution. You should indicate which parts of the definition of the Poisson process you use.

   (b) Suppose that 0.83, 0.30 and 0.57 are three values sampled from the uniform distribution on $(0, 1)$.

   Sample the first three event times from a Poisson process with rate $\lambda = 1/3$.

SECTION B

4. Let X be a random variable and let $F(x)$ be its cumulative distribution function. The generalised inverse of F is

$$F^{-1}(u) = \min\{x : F(x) \geq u\} \quad \text{for } 0 < u < 1.$$

(a) State the inverse transform algorithm for sampling a value for X.

Show that the algorithm works, i.e. that the cumulative distribution function of the values generated is F. You may use, without derivation, the cumulative distribution function of the uniform distribution on $(0, 1)$.

(b) Let the random variable X have probability density function

$$f(x) = \begin{cases} 2e^{2x}/(e^4 - e^2) & \text{for } x \in [1, 2] \\ 0 & \text{otherwise} \end{cases}$$

and suppose that 0.31 and 0.59 are two values sampled from the uniform distribution on $(0, 1)$. Sample two values from the distribution of X.

(c) Suppose that $X \sim \text{Bin}(3, 1/3)$, the binomial distribution with 3 trials and probability $1/3$ of success in each trial. Sketch the cumulative distribution function of X and its generalised inverse, paying careful attention to what happens at points of discontinuity. Write down the resulting algorithm for sampling a value of X using the inverse-transform method and explain how it relates to your sketch.

(d) Suppose that a biased coin with probability 0.6 of heads is tossed once. If the coin comes up heads, $X = 2$; otherwise X is sampled from $\text{Exp}(1/3)$.

Write down the cumulative distribution function of X. Deduce the inverse-transform algorithm for sampling values for X.

5. (a) State the *acceptance-rejection* algorithm for sampling from a continuous *target* probability distribution using values sampled from another continuous *proposal* probability distribution. Make clear where the acceptance and rejection takes place. State clearly any conditions needed on the two distributions.

   (b) Prove that the algorithm succeeds in sampling from the target distribution.
   State the acceptance rate of the algorithm.

   (c) The $\beta(a, b)$ distribution has probability density function

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \qquad \text{for } x \in (0, 1)$$

   where the parameters a and b are positive real numbers.

   (i) Show that the uniform distribution on $(0, 1)$ can be used as the proposal distribution to sample from $\beta(a, b)$ if $a \geq 1$ and $b \geq 1$ and find the optimal acceptance rate.

   (ii) Explain why it is not possible to use the uniform distribution on $(0, 1)$ as the proposal distribution if either parameter is less than 1.

   (iii) The symmetric triangular distribution on $(0, 1)$ has probability density function $2 - 4|x - \frac{1}{2}|$ for $x \in (0, 1)$ and which is zero otherwise.
   For what values of a can it be used as the proposal distribution to sample from $\beta(a, a)$? Justify your answer.

   (iv) In the context of deciding whether to use the triangular proposal distribution or the uniform proposal distribution when sampling from $\beta(a, a)$, explain the significance of the quantity $R = 4x'f(\frac{1}{2})/f(x')$ where $x' = (a-2)/(2a-3)$.