EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH3021-WE01

Title:
Differential Geometry III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. (a) Give the definition of a vertex of a regular plane curve. Find the vertices of the curve $\boldsymbol{\alpha}(u) = (u, \cos u)$, $u \in \mathbb{R}$.

   (b) State the 4-vertex theorem. For each assumption of the theorem explain what does it mean. Is there a curve satisfying all conditions of the 4-vertex theorem and having more than 4 vertices? Justify your answer.

2. Let $\boldsymbol{\alpha}(u) = (u^3 + u^2 + 3, u^2 - u + 1, u^2 + u + 1)$, $u \in \mathbb{R}$ be a curve in $\mathbb{R}^3$.

   (a) Show that $\boldsymbol{\alpha}$ is a regular curve.

   (b) Find the torsion τ of $\boldsymbol{\alpha}$ and determine whether the trace of $\boldsymbol{\alpha}$ is contained in a plane in $\mathbb{R}^3$.

3. (a) Give the definition of a regular surface in $\mathbb{R}^3$.

   (b) Show that the following sets are regular surfaces. State explicitly all statements you use in your proofs.

      (i) $S_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 3x^2 + 2y^2 - z^2 = 5\}$.

      (ii) $S_2 \subset \mathbb{R}^3$ given by $\boldsymbol{x}(r, \varphi) = (r \cos \varphi, r \sin \varphi, r^3)$, where $r > 0$, $0 \leq \varphi < 2\pi$.

4. Let S be the surface parametrised by $\boldsymbol{x}(u, v) = (u, v, u^3 + v^3)$.

   (a) Find the coefficients of the first and second fundamental forms.

   (b) Compute the Gauss curvature of S. Find the hyperbolic, flat and elliptic regions on S.

5. (a) Give the definition of an asymptotic curve on a surface. State the equivalent condition in terms of the second fundamental form.

   (b) Find all asymptotic curves on the surface $x^2 + y^2 = z$.

6. Consider the surface S parametrised by $\boldsymbol{x}(u, v) = (u, v, u^2)$.

   (a) Give the definition of a geodesic on a surface. Is the curve $\boldsymbol{\alpha}(w) = (w, 0, w^2)$ geodesic on S? Justify your answer.

   (b) Find at least one geodesic on S. Justify your answer.

SECTION B

7. A curve is called a *generalised helix* if its tangent makes a constant angle with a fixed line l in space. Such a line l is called an *axis* of the helix.

   (a) Let $\boldsymbol{\alpha}(u) = (e^u \cos u, e^u \sin u, e^u)$, $u \in \mathbb{R}$. Find curvature and torsion of $\boldsymbol{\alpha}$.

   (b) Show that the curve $\boldsymbol{\alpha}$ defined in part (a) is a generalised helix. Find its axis.

   (c) Can a generalised helix have two non-parallel axes? Justify your answer.

   (d) Let $f(s) : \mathbb{R} \to (0, \infty)$ be a smooth positive function. Show that there exists an arc length parametrised generalised helix $\beta(s)$ with curvature $\kappa(s) = f(s)$.

8. Define the first fundamental form on the upper half-plane $U = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$ by

$$E(u, v) := \frac{1}{v^2}, \qquad F(u, v) := 0, \qquad G(u, v) := \frac{1}{v^2}.$$

   (a) Consider the curve $\boldsymbol{\alpha}_{v_0}(t) \subset U$ given by $\boldsymbol{\alpha}_{v_0}(t) = (t, v_0)$ where $t \in (0, 1)$ and $v_0 > 0$ is a constant. Find the arc length $l(\boldsymbol{\alpha}_{v_0})$ of the curve $\boldsymbol{\alpha}_{v_0}(t)$. Find the limit of $l(\boldsymbol{\alpha}_{v_0})$ as $v_0 \to \infty$.

   (b) Consider the curve $\boldsymbol{\beta}(t) = (\frac{1}{2}, t) \subset U$, $t > 0$. Find the angle between $\boldsymbol{\alpha}_{v_0}$ and $\boldsymbol{\beta}$ at $\boldsymbol{p} \in U$, where $\boldsymbol{p} = \boldsymbol{\beta} \cap \boldsymbol{\alpha}_{v_0}$ is the intersection point of $\boldsymbol{\beta}$ and the curve $\boldsymbol{\alpha}_{v_0}$ defined in part (a).

   (c) Find the area of the domain $R_{u_0,v_0} = \{(u, v) \mid 0 < u < u_0, \;\; 0 < v_0 < v\} \subset U$.

   (d) Is the area of the domain $T = \{(u, v) \mid 0 < u < v < 1\} \subset U$ finite or infinite? *Hint:* Try to answer without any computations.

9. Let S be the surface given by the equation $x^2 + y^2 = z^2$, $z > 0$.

   (a) Show that $\boldsymbol{x}(r, \theta) = (r \cos \theta, r \sin \theta, r)$, where $r \in (0, +\infty)$, $\theta \in (-\pi, \pi)$ is a local parametrisation of S at the point $\boldsymbol{p} = (1, 0, 1)$.

   (b) Construct a local isometry at the point $\boldsymbol{p} = (1, 0, 1) \in S$ from S to a flat domain on the plane. You may use any parametrisation of S.

   (c) Find the geodesic on S passing through the point $\boldsymbol{p} = (1, 0, 1)$ and having the tangent vector $(0, 1, 0)$ at $\boldsymbol{p}$.

10. The regular surface $S \subset \mathbb{R}^3$ is given by

$$S = \{(x, y, z) \in \mathbb{R} \mid x^2 + y^2 - z^{2/3} = 0, \;\; 1 < z < 8\}.$$

   (a) Find the value of the integral $\int_S K dA$.

   (b) State the global Gauss-Bonnet Theorem explaining all notions which you use.

   (c) Find the Euler characteristic of S.

   (d) Verify the global Gauss-Bonnet Theorem directly for the surface S.