Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH3041-WE01

Title: Galois Theory III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. Let K be a splitting field for $X^7 - 5$ over $\mathbb{Q}$.
   - (a) Prove that K contains a primitive 7-th root of unity.
   - (b) Find the degree $[K : \mathbb{Q}]$ of K over $\mathbb{Q}$. (Justify your answer.)
2. Let $P = X^4 + X + 1 \in \mathbb{F}_2[X]$.
   - (a) Prove that P is irreducible.
   - (b) Let $K = \mathbb{F}_2(\theta)$, where θ is a root of P. Apply the Kummer theory to prove that $X^5 - \theta \in K[X]$ is irreducible.
3. Suppose L/K is a finite Galois extension and E is a field between K and L.
   - (a) Prove that L/E is Galois.
   - (b) Is E/K always Galois? (Justify your answer.)
   - (c) Prove that if $[E : K] = 2$ then E/K is Galois.
4. (a) Give the definition of separable field extension.
   - (b) Give an example of a separable quadratic extension L of a field K of characteristic 2.
   - (c) Give an example of an inseparable quadratic extension E of a field F of characteristic 2.
5. Suppose $\zeta_{13} \in \mathbb{C}$ is a 13-th primitive root of unity.
   - (i) Prove the existence of a unique subfield K of $\mathbb{Q}(\zeta_{13})$ such that $[\mathbb{Q}(\zeta_{13}) : K] = 2$.
   - (ii) Find an element $\theta \in K$ such that $K = \mathbb{Q}(\theta)$. (Justify your answer.)
6. Find the Galois groups of the following polynomials in $\mathbb{Q}[X]$:
   - (a) $X^3 - 3X + 2$;
   - (b) $X^3 - 3X - 1$.

SECTION B

7. Suppose L is a splitting field for $X^4 - 7 \in \mathbb{Q}[X]$.

   (a) Describe the structure of $G = Gal(L/\mathbb{Q})$.

   (b) Find all subfields K of L such that $[K : \mathbb{Q}] = 2$. (Justify your answer.).

   (c) Find all subfields E of L such that $[E : \mathbb{Q}] = 4$. (Justify your answer.)

   (d) Prove that $\sqrt[4]{5} \notin \mathbb{Q}(\sqrt[4]{7})$.

8. Let L be a splitting field for $(T^3 - 1)(T^{11} - 1)(T^{19} - 1) \in \mathbb{Q}[T]$.

   (a) Find the degree $[L : \mathbb{Q}]$ and the Galois group $Gal(L/\mathbb{Q})$.

   (b) Prove that L does not contain $\sqrt{5}$.

   (c) Find a subfield E in L such that $[E : \mathbb{Q}] = 8$.

   (d) Prove that the above subfield E is unique, i.e. if K is a subfield of L such that $[K : \mathbb{Q}] = 8$ then $K = E$.

9. Let L be a splitting field for $X^4 + 2X^2 - 4 \in \mathbb{Q}[X]$.

   (a) Find the Galois group of L over $\mathbb{Q}$.

   (b) Prove that $\mathbb{Q}(\sqrt{5}, i) \subset L$.

   (c) Find the Galois group of L over $\mathbb{Q}(\sqrt{-5})$.

   (d) Find the Galois group of L over $\mathbb{Q}(i)$.

10. (i) Find all complex roots of the polynomial (where $i = \sqrt{-1}$)

$$X^4 - 3X^2 + 2\sqrt{3}iX + 1\,.$$

   (ii) (a) List all subfields of $\mathbb{F}_{81}$.

   (b) Does this field contain a primitive 16-th root of unity? (Justify your answer.)

   (c) Find the degrees of all irreducible polynomials dividing $X^{80} - 1 \in \mathbb{F}_3[X]$.