Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH3071-WE01

Title: Decision Theory III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. (a) Rachel is considering whether to play a dice game at a festival. The initial stake is 2 pounds, and it doubles every time an odd number is rolled on a fair 6-sided die (i.e. with equally likely possible outcomes $\{1, 2, 3, 4, 5, 6\}$). If an even number is rolled, the game ends and Rachel takes all the money accummulated so far. Write down the gamble associated with this game and calculate its corresponding Expected Money Value (EMV).

   (b) Show that maximizing EMV in this scenario leads to irrational behaviour.

   (c) Assume that Rachel's utility function for money is given by $U(£x) = \sqrt{x}$, $x \geq 0$. What is Rachel's utility for the gamble in item (a)?

   (d) What is the certainty equivalent and risk premium associated with this gamble? Comment on Rachel's risk behaviour and suggest a rational strategy to decide whether to play the game or not.

2. Sam is at a cake shop but is having a hard time deciding what to get. The options are chocolate, lemon, vanilla, fruit, or carrot cake. The shop owner is starting to get impatient and proposes the following game. He will flip a fair coin two times. If it lands heads twice, Sam will have chocolate cake. If it lands tails followed by heads, he will have lemon cake. If it lands heads followed by tails, he will have vanilla cake. And if it lands tails twice, he will have fruit cake. However, Sam can also choose to not wait for the coin flips and take carrot cake for sure.

   (a) Draw a decision tree and an influence diagram to represent this problem.

   (b) List the necessary and sufficient conditions for solving an influence diagram.

   (c) Solve the influence diagram in item (a) explaining all steps taken.

3. Let $\mathcal{R}$ be a set of basic rewards, $G(\mathcal{R})$ be the set of all gambles over $\mathcal{R}$, and U be a utility function modelling a preference ordering $\succcurlyeq^*$ on $G(\mathcal{R})$.

   (a) Show that

$$U(p_1 r_1 \oplus \ldots \oplus p_k r_k) = \sum_{i=1}^{k} p_i U(r_i),$$

   for all $k \geq 1$, all $p_1, \ldots, p_k \geq 0$ with $\sum_{i=1}^{k} p_i = 1$, and all basic rewards $r_1, \ldots, r_k \in \mathcal{R}$.

   (b) Show that $V = aU + b$ is a utility function that also models the same preference ordering $\succcurlyeq^*$ for $a, b \in \mathbb{R}$, $a > 0$.

4. The procedure of an organisation to select a new chairperson out of 6 candidates consists of two stages. In Stage 1, all voters provide their preference order over the candidates for the position. A scoring system is used where, for each voter, their j-th preference gets $7 - j$ points, for $j = 1, \dots, 6$. The score for each candidate is defined as the sum of all voters' points for them. The two candidates with the highest scores are only considered in Stage 2, but if there is a tie then this stage may involve more than two candidates. The candidates who do not make it into Stage 2 are ranked according to their scores in Stage 1.

   In Stage 2 the two (or more, in case of a tie) candidates with highest scores in Stage 1 are considered. Now a simple majority rule is used to rank these candidates, based on the voters' original preferences. If this procedure does not lead to a single person being selected, those candidates with the highest number of votes in Stage 2 will be ranked equally and jointly appointed as chairpersons.

   There are 20 voters, whose preferences over candidates A to F are provided in the table below.

number of voters:	2	6	3	5	4
1st	A	C	F	A	F
2nd	B	F	E	B	A
3rd	C	D	D	D	B
4th	D	A	C	C	E
5th	E	E	B	E	D
6th	F	B	A	F	C

   (a) Derive the group preference order over the six options.

   (b) Discuss this general two stage selection procedure from the perspective of Arrow's theorem. Include a description of the four axioms in this theorem and explain for each of these axioms whether or not they are satisfied by this procedure.

5. In a particular game, R chooses strategy R1 or R2, C chooses strategy C1, C2, C3, C4 or C5. The payoffs to R are as follows

	$C1$	$C2$	$C3$	$C4$	$C5$
$R1$	-1	0	2	5	3
$R2$	4	2	1	0	1

   The payoff to C is minus the payoff to R.

   (a) Explain what a 'dominated strategy' is and identify any such strategies in this game.

   (b) Use a graphical method to identify the minimax strategies for R and for C, and the value of the game.

   (c) Explain briefly why the analysis of two-person non-cooperative games, where the sum of the payoffs to the two players is not constant, may be more complicated than for zero-sum games.

6. (a) Consider the decision table with utilities given below, with four possible decisions a_i, $i = 1, 2, 3, 4$. Determine the optimal decision according to each of the following optimality criteria: (1) Maximum expected utility, assuming the probability for state θ_j to be the true state is $P(\theta_j)$, as given in the table; (2) maximisation of minimum utility; (3) Minimisation of maximum regret.

	θ_1	θ_2	θ_3	θ_4
$P(\theta_j)$	0.4	0.3	0.2	0.1
a_1	0	2	-3	14
a_2	2	2	2	2
a_3	-2	3	4	6
a_4	6	-4	-1	10

(b) In the theory of two-person zero-sum games, as discussed in the lectures, it is assumed that each player aims at maximisation of minimum utility ('maximin criterion'). Briefly explain why this may be an attractive criterion to use in such games. Suppose that, in such a game, a player instead aims at maximisation of expected utility; briefly explain if this would make the analysis of the game, in order to derive an optimal strategy, easier or harder compared to the use of the maximin criterion.

SECTION B

7. The Swedish Chef loves making meatballs. The worst acceptable recipe calls for 4 meatballs of 50 grams each, and it would be unacceptable to make a dish with fewer meatballs or less meat than that. The Chef thinks that the number of meatballs (n) and their individual weight (w) are mutually independent attributes. He also believes that there is no such thing as too many meatballs, and his marginal utilities for the number of meatballs and their size are non-decreasing.

   His marginal utilities are of the form:

$$U(4, w) = \alpha + \beta w^2$$
$$U(n, 50) = \gamma + \delta\sqrt{n}.$$

   The Chef is adamant that $(0, 50) \sim^* (4, 0)$, and he also says that a recipe containing 4 meatballs of 500 grams each is equivalent to a recipe with 16 meatballs of 125 grams each.

   (a) Find the Chef's joint utility as a function of the number of meatballs and their weight. Specify if the Chef finds these attributes substitutable or not.

   (b) The Chef has changed his mind regarding his marginal utilities. He now thinks that $U(0, w) = 40w - w^2$, and specifies the following isopreference curve over the number of meatballs and their weight:

$$n \cdot w = 10.$$

   He also says that 12 meatballs of 20 grams are equivalent to 10 meatballs of 25 grams. Find the Chef's new utility function for number of meatballs and weight. Determine whether the Chef's attitude towards these attributes' substitutability has changed.

8. Dr Bunsen Honeydew finds out that his assistant Beaker is very ill. Without further treatment, Beaker is likely to die in 3 months. Dr Honeydew comes up with a treatment involving a very risk operation. Beaker is expected to live about 1 year if he survives the operation. However, the probability of Beaker dying during this operation is 0.2.

   Dr Honeydew discovers a new test that will provide uncertain information about whether Beaker would survive the operation or not. When positive, the odds of Beaker surviving the operation increase. It is known that patients that survive this operation have a positive test result 85% of the time. However, patients that die as a result of this operation have a positive result 2% of the time.

   (a) What is Beaker's probability of surviving the operation if the test is positive?

   (b) Construct and solve the decision tree for the decision problem as to whether Beaker should go through the operation or not.

   (c) We then find out that Dr Honeydew's test may have some fatal complications. That is, Beaker might die during the test with probability 0.005. Update the decision tree in item (b), find the optimal solution and its expected utility.

9. Adam and Eve are planning to celebrate their forthcoming retirement. The options they consider are a party with their friends, a holiday in a nice hotel, and a cruise. Their utilities for each option are given in the following table.

	Party	Hotel	Cruise
Adam	2	6	5
Eve	6	2	5

As status quo option they consider not to have any celebration, for which Adam has utility 0 and Eve has utility 1.

(a) Sketch the feasible region for this problem, identify the Pareto Boundary and the status quo point.

(b) Find the Nash point and the equitable distribution point for this bargaining problem. What do you recommend Adam and Eve to do?

(c) A friend asks if they have considered adding a conservatory to their house now that they will spend more time at home, and this turns out to be possible given their budget. Adam likes this option and assigns utility 6 to it. Eve has yet to think about this option and her utility for it. Let Eve's utility for a conservatory be denoted by real-valued V. Derive the Nash point as a function of V.

(d) In the lectures, the Nash point and the equitable distribution point have been presented as possible solutions to such bargaining problems. Discuss briefly a property of the Nash point which one may consider to be a disadvantage, and which is less of a problem for the equitable distribution point.

10. Harsanyi's theory of 'utilitarianism' considers the following scenario for group decisions. Each individual in a group ('citizen') expresses utility for each available option. Suppose m citizens face r social choices $x_1, x_2, \ldots, x_r$. Each citizen is 'rational', so citizen i has a utility function $U_i(x)$ over these choices. In addition to these r options, we assume that one option can be introduced, say x_0, for which all citizens agree that it would be as bad as what they consider the worst real option. Each citizen's utility function U_i is scaled to lie in $[0, 1]$, with citizen i's utility equal to 0 for the least liked option, as well as for x_0, and equal to 1 for the most preferred option.

(a) State Harsanyi's theorem on the combination of the citizens' utilities when aiming at a good choice for the community. You should introduce in detail the two axioms of 'social rationality' on which this theorem is based.

(b) Prove Harsanyi's theorem for the case of $m = 2$ citizens.

(c) Parents of 4 children (A, B, C, D) want to decide where to go for a mid-term holiday, to make their children happy. The options are Disneyland Paris (DP), Peppa Pig World (PP), CBeebies Land (CB) or a city trip to London (Lo). The utilities of the children for these options are given in the table below. All children assign utility 0 to the additional option of not going on holiday. Apply Harsanyi's theorem to rank these options. Suppose that children C and

D discover the utilities of children A and B before stating their own utilities; explain whether or not this provides children C and D the opportunity to jointly manipulate the overall result.

	DP	PP	CB	Lo
A	0.2	0.8	1	0
B	0.4	1	0.7	0
C	1	0.8	0	0.4
D	0.6	0	0	1