EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH3091-WE01

Title:
Dynamical Systems III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. A three-dimensional dynamical system is governed by the differential equations

$$\begin{aligned}\dot{x}_1 &= -x_2 + x_3 + x_1 x_2 \\ \dot{x}_2 &= -x_1 + x_3 + x_1^2 \\ \dot{x}_3 &= -2x_2 + x_3 + 2x_2^2 \ .\end{aligned}$$

   (a) Determine the critical points of this system.

   (b) Show that the plane $x_1 = x_2$ is an invariant set.

   (c) Does the plane $x_1 = x_2$ contain any critical points around which the linearized equations (restricted to the $x_1 = x_2$ plane) have a non-vanishing center manifold?

2. Consider the second order ordinary differential equation

$$\ddot{x} - \lambda \dot{x} - (\lambda - 1)(\lambda - 2)x = 0$$

   for the function $x(t)$. λ is a real parameter.

   (a) Set $\dot{x} = y$. Find the critical points of the resulting two-dimensional dynamical system. Make sure to analyse carefully the equations at all possible values of λ.

   (b) Characterize the critical points as sources, sinks etc. Can the nature of the critical points change as λ is varied?

   (c) Sketch the flow in the two-dimensional phase space for $\lambda = 3$.

3. An n-dimensional autonomous dynamical system has the form

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$$

   where $\boldsymbol{x}(t)$ and $\boldsymbol{F}$ are vectors in $\mathbb{R}^n$.

   (a) Write the defining equations of the Picard iteration scheme for this system with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

   (b) Let $\boldsymbol{F}(\boldsymbol{x}) = A\boldsymbol{x}$, where A is an $n \times n$ constant matrix. What is the limit of the Picard sequence in this case?

   (c) Consider a two-dimensional case where $\boldsymbol{x} = \binom{x_1}{x_2}$ and

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{pmatrix} x_1 + 3x_2 + x_2^2 \\ 2x_1 - x_1^2 \end{pmatrix} \ .$$

   The initial state is $\boldsymbol{x}(0) = \binom{1}{0}$. Compute the first three members of the Picard sequence: $\boldsymbol{x}^{(0)}(t), \boldsymbol{x}^{(1)}(t), \boldsymbol{x}^{(2)}(t)$.

4. (a) State a necessary condition for a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ to undergo a local bifurcation at a point $\boldsymbol{x}^*$.

 (b) Consider the Lorenz system

$$\dot{x} = \sigma(y - x) \qquad \dot{y} = rx - y - xz \qquad \dot{z} = xy - bz$$

 Let us fix $b = 1$. Identify all the points (r, σ) where the necessary condition is obeyed for a local bifurcation at the origin $\boldsymbol{x}^* = 0$. (For this exercise the parameters r and σ can take arbitrary real values.)

 (c) Give the normal forms of a *transcritical* and a *pitchfork* bifurcation in a one-dimensional dynamical system.

5. (a) Define what is meant by two dynamical systems to be topologically conjugate.

 (b) Consider two n-dimensional dynamical systems $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ and $\dot{\boldsymbol{y}} = \boldsymbol{G}(\boldsymbol{y})$ with a finite number of fixed points. Suppose that they are topologically conjugate on all of $\mathbb{R}^n$. Prove that they have an equal number of fixed points.

 (c) Prove that the one-dimensional systems $\dot{x} = x(x-1)$ and $\dot{x} = x^2 + 3x + 4$ are not topologically conjugate on all of $\mathbb{R}$.

 (d) State but do not prove the Hartman-Grobman theorem.

6. Consider the planar dynamical system

$$\dot{x} = -\frac{1}{2}x + y + \frac{1}{2}(x^3 + y^2 x) \qquad \dot{y} = -x - \frac{1}{2}y + \frac{1}{2}(y^3 + x^2 y)$$

 (a) Transform the system into polar coordinates.

 (b) State the conditions for a function $V(x, y)$ to be a Lyapunov function for the fixed point at the origin.

 (c) Find a Lyapunov function for the fixed point at the origin.

 (d) Is the system Hamiltonian? Explain your answer.

SECTION B

7. The two-dimensional dynamical system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{x}) \tag{1}$$

with

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ , \quad A = \begin{pmatrix} -3 & -2 \\ 2 & 2 \end{pmatrix} \ , \quad \boldsymbol{B}(\boldsymbol{x}) = \begin{pmatrix} x_2^2 \\ x_1^2 + x_1 x_2 \end{pmatrix}$$

has a critical point at $\boldsymbol{x} = 0$.

(a) Show that the linearized system around $\boldsymbol{x} = 0$ is $\dot{\boldsymbol{x}} = A\boldsymbol{x}$. Solve this linear system. What are the dimensions of the stable, unstable and center manifold of the linear system? Is the point $\boldsymbol{x} = 0$ a hyperbolic fixed point?

(b) Find a linear transformation $\boldsymbol{x} = M\boldsymbol{y}$ that diagonalizes the matrix A. In the $\boldsymbol{y}$ coordinates the non-linear system (1) takes the form

$$\dot{\boldsymbol{y}} = \tilde{A}\boldsymbol{y} + \tilde{\boldsymbol{B}}(\boldsymbol{y}) \ . \tag{2}$$

Determine the 2×2 diagonal matrix $\tilde{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and the vector $\tilde{\boldsymbol{B}}(\boldsymbol{y})$.

(c) Let $U(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & 0 \end{pmatrix}$ and $V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$. Show that the solutions of the integral equation

$$\boldsymbol{u}(t, \boldsymbol{a}) = U(t)\boldsymbol{a} + \int_0^t ds\, U(t-s)\tilde{\boldsymbol{B}}(\boldsymbol{u}(s, \boldsymbol{a})) - \int_t^\infty ds\, V(t-s)\tilde{\boldsymbol{B}}(\boldsymbol{u}(s, \boldsymbol{a})) \tag{3}$$

are also solutions of the non-linear equation (2) for arbitrary vector $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. Is $\boldsymbol{u}(t, \boldsymbol{a})$ sensitive to the value of the component a_2?

(d) The integral equation (3) can be solved with the Picard iteration method. Set $\boldsymbol{u}^{(0)} = 0$. Evaluate the next 2 members of the Picard sequence: $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$.

(e) Express the component $y_2 = u_2^{(2)}(0, \boldsymbol{a})$ of the Picard iterate $\boldsymbol{u}^{(2)}$ at $t = 0$ as a function of the component $y_1 = u_1^{(2)}(0, \boldsymbol{a})$ of the same Picard iterate. The curve $y_2 = y_2(y_1)$ is expected to be a good approximation to the stable manifold of the non-linear system (2) appropriately close to the fixed point $\boldsymbol{y} = 0$. Compare the curve $y_2 = y_2(y_1)$ with the curve of the stable manifold in the linear system $\dot{\boldsymbol{y}} = \tilde{A}\boldsymbol{y}$. Is one tangent to the other at the critical point in agreement with the stable manifold theorem?

8. Consider the second order differential equation $\ddot{x} + f(x) = 0$ for the function $x(t)$. $f(x)$ is a given C^1 function.

   (a) Write this equation as a dynamical system by setting $\dot{x} = y$. Prove that the quantity
   $$F(x, y) = \frac{1}{2}y^2 + \int^x ds\, f(s)$$
   is a first integral of this system.

   (b) Use this general result to find a first integral for the case $f(x) = \sin x$ which describes the mathematical pendulum. Deduce an equation for the trajectories of the pendulum in phase space. Determine the critical points of the system and perform a linear analysis around them. Draw the phase flow.

   (c) With similar methods derive a first integral for the case $f(x) = x - \frac{1}{2}x^2$ and use the first integral to determine an equation for the trajectories of the dynamical system. Determine the critical points and perform a linear analysis around them. Draw the phase flow.

9. Consider the planar dynamical system
   $$\dot{x} = 2(y + x)^4 - h(x, y)^3 \qquad \dot{y} = -2(y + x)^4 - (y + x)^3 + h(x, y)^3$$
   with $h(x, y) = x + (x + y)^2$.

   (a) Show that the system has only one fixed point which is the origin.

   (b) State the Poincaré-Bendixson theorem.

   The above dynamical system contains a positively invariant compact domain which contains an open neighbourhood of the origin. Let us call it D.

   (c) As a consequence of the Poincaré-Bendixson theorem, there are the three possible ω limit sets for any point inside D. One of them is a periodic orbit. State the other two possibilities.

   (d) Give Bendixson's criterion and its proof.

   (e) Use Bendixon's criterion to show that the system has no periodic orbits whatsoever, so the "periodic orbit" option is not realised in this system.

   (f) Adapt the proof of Bendixson's criterion to rule out one of the other two possibilities you stated under point (c) as well.

   (g) Is the fixed point asymptotically stable? Explain your answer.

10. (a) State the flow box theorem.

    (b) State the definition of the Poincaré index for a C^1 planar dynamical system $(\dot{x}, \dot{y}) = (f(x, y), g(x, y))$.

Consider a three-dimensional dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ for which $H(x, y, z) = x^2 + y^2 - z^2$ is a first integral. Let D be the solid cylinder of unit radius, i.e.

$$D = \{(x, y, z) | x^2 + y^2 < 1\}$$

    (c) You are given that, at every point on the boundary of D with $x^2 + y^2 = 1$, the flow points outwards from D. Show that you can use the Poincaré index to argue that there is a fixed point in the dynamical system on each of the surfaces given by $H(x, y, z) = c$ with $c < 0$.

The next questions pertain to the specific three-dimensional dynamical system given by

$$\dot{x} = xz^2 \qquad \dot{y} = yz^2 \qquad \dot{z} = x^2 z + y^2 z$$

    (d) Transform the system to cylindrical coordinates.

    (e) Find the omega limit set for every point in $\mathbb{R}^3$.

    (f) State the definition of an absorbing set. Does the given system have an absorbing set? Briefly explain your answer.