Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH3101-WE01

Title:
Continuum Mechanics III

Time Allowed:	3 hours	
Additional Material provided:	Formula sheet	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. The velocity vector field $\boldsymbol{v}(r,\theta,z) = r\boldsymbol{e}_r$ of a continuous medium is given in cylindrical coordinates (r,θ,z), $r \in [0,\infty)$, $\theta \in [0,2\pi]$, $z \in \mathbb{R}$, where

$$\begin{cases} x^1 = r\cos\theta \\ x^2 = r\sin\theta \\ x^3 = z \end{cases}$$

   (a) Sketch the particle paths corresponding to the vector field $\boldsymbol{v}$;

   (b) Write down the continuity equation satisfied by the vector field $\boldsymbol{v}$ in cylindrical coordinates.

2. The complex potential $w(z)$ is given by

$$w(z) = \frac{1}{z^2}, \ z \in \mathbb{C} \setminus \{0\},$$

   where $z = x + iy$, $x,\ y \in \mathbb{R}$, $x^2 + y^2 \neq 0$.

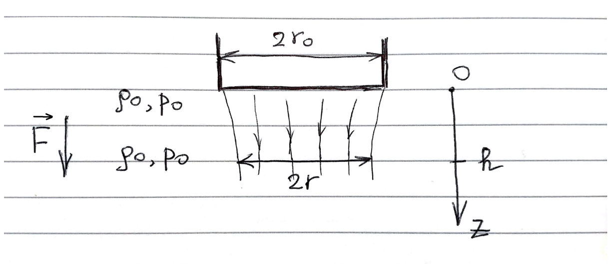
   (a) Find the velocity potential $\phi(x,y)$ and the stream function $\psi(x,y)$ corresponding to the complex potential $w(z)$;

   (b) Find the velocity field $\boldsymbol{u} = (u^1(x,y), u^2(x,y))$ corresponding to the complex potential $w(z)$.

   (c) Find the flux $\int_\gamma \boldsymbol{u}.\boldsymbol{n}\, dl$ of the velocity field $\boldsymbol{u}$ corresponding to $w(z)$ across the curve $\gamma = \{(s,\ s),\ s \in [1,2]\}$ connecting the points $A = (1,1)$ and $B = (2,2)$. The unit normal vector $\boldsymbol{n}$ to the curve γ is chosen in such way that it points in the negative imaginary direction.

3. (a) Let $\boldsymbol{v}(x) = (v^1(x), v^2(x), v^3(x))$ and $p = p(x)$, $x \in \mathbb{R}^3$, be smooth and represent the velocity vector field and pressure of a steady incompressible ideal fluid with constant density $\rho(t,x) \equiv \rho_0 > 0$. Assuming that the external force takes the form $\boldsymbol{F}(t,x) = -\nabla\mathcal{V}(x)$, where $\mathcal{V}(x)$ is a smooth scalar function, write down the corresponding Bernoulli integral.

   (b) A vertical jet of ideal incompressible fluid flows out of a round tap of radius r_0 with absolute velocity $v_0 > 0$. Assuming that the flow is steady, the density and the pressure are constant ($\rho \equiv \rho_0 > 0$ and $p \equiv p_0 > 0$) and the only external force is gravity $\boldsymbol{F}(t,x) = (0,0,g)$, find the radius r of the jet at the distance h below the tap (see the figure below).
   Hint: use the Mass Conservation Law and the theorem on the Bernoulli integral.



4. Consider an isotropic Newtonian fluid such that the deviatoric stress tensor is

$$d_{ij} = A_{ijkl}\partial_k u_l \,,$$

for some coefficients A_{ijkl}.

(a) Show that the most general form of a 4th rank isotropic tensor is:

$$A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} \,.$$

(b) Using the fact that the stress tensor is defined as $\sigma_{ij} = -p\delta_{ij} + d_{ij}$ and it has the property $\sigma_{ij} - \sigma_{ji} = 0$, determine the relations between α, β and γ.

(c) Give the physical interpretation of the constant β .

5. (a) What is a Stokes flow?

(b) Starting from the dimensionless Navier-Stokes equations for an incompressible fluid

$$\partial_{t'}\boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla'\boldsymbol{u}' = -\frac{P}{U^2\rho_0}\nabla' p' + \frac{1}{Re}\Delta'\boldsymbol{u}' \,,$$
$$\nabla \cdot \boldsymbol{u}' = 0,$$

deduce the equations for a Stokes flow.

(c) For given viscosity and length-scale, is a Stokes flow more likely to appear at high or low velocities? Justify your answer.

6. (a) Give the mathematical expressions of the phase and group velocities and state an example of waves where the group velocity is the same as the phase velocity.

(b) Consider water waves with dispersion relation

$$\omega(k) = \sqrt{gk\tanh(kH)},$$

where g is the gravitational acceleration and H is the depth of the ocean.
Show that the group velocity is

$$c_G = \frac{c}{2}\left[1 + \frac{2kH}{\sinh(2kH)}\right],$$

where c is the phase velocity.

(c) Taking the appropriate limits of H in question (b), find whether or not (i) deep and (ii) shallow water waves are dispersive.

SECTION B

7. Let $\boldsymbol{v}(x) = (v^1(x), v^2(x), v^3(x))$, $x \in \mathbb{R}^3$, be a vector field given in Cartesian coordinates $x = (x^1, x^2, x^3)$ such that

$$\begin{cases} \operatorname{div} \boldsymbol{v}(x) = \theta(x), \\ \operatorname{rot} \boldsymbol{v}(x) = 0, \end{cases} \qquad \theta(x) = \begin{cases} |x|^2 - R^2, & |x| \le R, \\ 0, & |x| > R, \end{cases}$$

where $R > 0$ is a constant and $|x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

(a) Compute the cubic potential

$$V(x) = \int_{\mathbb{R}^3} \frac{\theta(y)}{|x-y|}\, dy;$$

Hint: at the end of the calculation one needs to consider two regions separately: $|x| \le R$ and $|x| > R$.

(b) Using the obtained expression for $V(x)$ find the vector field $\boldsymbol{v}(x)$;

(c) Assuming that the vector field $\boldsymbol{v}(x)$ is velocity field of an ideal incompressible fluid with a constant density, $\rho(x) \equiv \rho_0 > 0$, and the external force is 0, find the pressure $p(x)$. We suppose that the pressure $p(x)$ is continuous and satisfies the extra condition $\lim\limits_{|x| \to +\infty} p(x) = 0$.

8. (a) Let $D \subset \mathbb{R}^3$ be a smooth bounded domain. Let also $\Big(\boldsymbol{v}(t,x), p(t,x)\Big)$, where $x \in D$, $\boldsymbol{v}(t,x) = (v^1(t,x), v^2(t,x), v^3(t,x))$, $p(t,x)$ is a scalar function, be a smooth solution to the incompressible Euler's equations:

$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v}, \nabla)\boldsymbol{v} + \frac{\nabla p}{\rho_0} = -\nabla \mathcal{V}, \ x \in D, \ t \ge 0, \\ \operatorname{div} \boldsymbol{v} = 0, \ x \in D, \ t \ge 0, \\ \boldsymbol{v}.\boldsymbol{n}|_{\partial D} = 0, \ t \ge 0. \end{cases}$$

Here $\rho_0 > 0$ is a constant, $\mathcal{V} = \mathcal{V}(t,x)$ is a smooth scalar function, $\boldsymbol{n} = \boldsymbol{n}(t,x)$ is a unit outward normal vector to the boundary ∂D. Write down the evolution equation for the vorticity $\boldsymbol{\omega}(t,x) = \operatorname{rot} \boldsymbol{v}(t,x)$;

(b) Under the assumptions of part (a), prove that

$$\frac{d}{dt} \int_D \boldsymbol{\omega}(t,x)\, dx = \int_{\partial D} \boldsymbol{v}\ \boldsymbol{\omega}.\boldsymbol{n}\, d\sigma;$$

Remark: the integral of a vector field $\boldsymbol{u} = (u^1(x), u^2(x), u^3(x))$, $x \in D$, is understood in the natural way

$$\int_D \boldsymbol{u}(x)\, dx = \begin{pmatrix} \int_D u^1(x)\, dx \\ \int_D u^2(x)\, dx \\ \int_D u^3(x)\, dx \end{pmatrix}, \qquad \int_{\partial D} \boldsymbol{u}\, d\sigma = \begin{pmatrix} \int_{\partial D} u^1\, d\sigma \\ \int_{\partial D} u^2\, d\sigma \\ \int_{\partial D} u^3\, d\sigma \end{pmatrix}.$$

Hint: we can put the time derivative under the sign of the integral.

(c) Let $D \subset \mathbb{R}^3$ be a smooth bounded domain. Prove that for a smooth vector field $\boldsymbol{u}(x) = (u^1(x), u^2(x), u^3(x))$, $x \in \mathbb{R}^3$, such that $\boldsymbol{u}|_{\partial D} = 0$, the following equality holds

$$\int_D \left(|\operatorname{div} \boldsymbol{u}(x)|^2 + |\operatorname{rot} \boldsymbol{u}(x)|^2\right) dx = \int_D \left(|\nabla u^1(x)|^2 + |\nabla u^2(x)|^2 + |\nabla u^3(x)|^2\right) dx.$$

Hint: for a smooth function $f(x)$, $x \in \mathbb{R}^3$ we have $\partial_{x^i}\partial_{x^j} f(x) = \partial_{x^j}\partial_{x^i} f(x)$, where $i \neq j$ and $i, j \in \overline{1,3}$.

9. Consider two coaxial cylinders of radii R_1 and R_2 with $R_1 < R_2$, whose common axis is parallel to $\mathbf{e}_z$. The cylinders are initially at rest. The volume between the two cylinders is filled with an incompressible fluid of density $\rho = 1$ and viscosity μ.

(a) Assuming that $\boldsymbol{u} = u_z(r)\boldsymbol{e}_z$, find the steady-state solution describing the flow of the fluid if the pressure is $p = -Gz$, for some constant G.

(b) Once the fluid has reached the steady-state described in part (a) the outer cylinder starts rotating at angular velocity Ω. After some time the fluid relaxes to a new steady-state $\boldsymbol{u} = u_\theta(r)\boldsymbol{e}_\theta + u_z(r)\boldsymbol{e}_z$, where θ is the azimuthal coordinate. Find this steady flow.

(c) Find the energy dissipation rate of the flow per unit length of the cylinder for the steady-state flow corresponding to part (a).

10. Consider the shallow water equations:

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -g\nabla h$$

and

$$\partial_t h + \nabla \cdot (h\boldsymbol{v}) = 0\,,$$

where $\boldsymbol{v}$ is equal to (u_x, u_y) and h is the total water depth.

(a) Show that the shallow water equations conserve both

$$E = \frac{1}{2}\int_D \left(h|\boldsymbol{v}|^2 + gh^2\right) dV$$

and

$$Z = \int_D \frac{(\nabla^\perp \cdot \boldsymbol{v})^2}{h} dV\,.$$

(b) Linearise the shallow water equations by setting $\boldsymbol{v} = \hat{\boldsymbol{v}}$ and $h = H + \hat{\eta}$. Determine the wave speed for $\hat{\eta}$.

(c) When the Coriolis term is included the shallow water equations are modified so that

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\Omega \boldsymbol{e}_z \times \boldsymbol{v} = -g\nabla h$$

and

$$\partial_t h + \nabla \cdot (h\boldsymbol{v}) = 0\,,$$

where Ω is a constant. Linearise these equations and find the dispersion relation for waves of the form $\hat{\boldsymbol{v}} = \tilde{\boldsymbol{v}} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$ and $\hat{\eta} = \tilde{\eta} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$.