EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH3111-WE01

Title:
Quantum Mechanics III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. The Hilbert space of a Quantum Mechanical system has an orthonormal basis $B = \{|a\rangle, |b\rangle\}$. The linear operator

$$\hat{M} = i\ |a\rangle\langle b|\ + c\ |b\rangle\langle a|\ ,$$

with c a complex constant, is a physical observable.

   (a) Fix the constant c.

   (b) Find the matrix form of $\hat{M}$ in the basis B.

   (c) What are the possible outcomes in a measurement of $\hat{M}$?

   (d) Find the expectation value $\langle\, \hat{M}\,\rangle$ for the state $|\psi\rangle = (|a\rangle + i\ |b\rangle)/\sqrt{2}$.

2. (a) If $\hat{x}$ and $\hat{p}$ are the position and momentum operators of a Quantum Mechanical particle, use induction to show that $[\hat{x}^n, \hat{p}] = i\, n\, \hbar\, \hat{x}^{n-1}$ for positive integers n.

   (b) If $f(x)$ is an analytic function, compute the commutator $[\hat{f}(\hat{x}), \hat{p}]$.

   (c) For a particle in two dimensions with position operators $\hat{x}$ and $\hat{y}$ and corresponding momentum operators $\hat{p}_x$ and $\hat{p}_y$, the angular momentum operator is $\hat{L} = \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x$. Compute the commutator $[\hat{L}, \hat{f}(\hat{x}^2 + \hat{y}^2)]$ where $f(z)$ is an analytic function.

3. (a) If $\hat{H}$ is the Hamiltonian of a Quantum Mechanical system, state Schrödinger's equation in Dirac's notation.

   (b) Show that for any time independent observable $\hat{\Omega}$ the expectation value $\langle\, \hat{\Omega}\, \rangle$ satisfies

$$\hbar \frac{d}{dt}\langle\, \hat{\Omega}\, \rangle = \langle\, i\,[\hat{H}, \hat{\Omega}]\, \rangle\, .$$

   (c) The Hamiltonian of a Quantum system is $\hat{H} = \varepsilon_0\,(\hat{a}\,\hat{b} + \hat{b}\,\hat{a})$ with $\hat{a}$, $\hat{b}$ Hermitian operators satisfying $[\hat{a}, \hat{b}] = i\,\hbar$ and ε_0 a real number. Determine the time dependence of $\langle\, \hat{a}\, \rangle$ and $\langle\, \hat{b}\, \rangle$.

4. Consider the three self-adjoint operators $\hat{J}_i$, $i \in \{1, 2, 3\}$ obeying

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\epsilon_{ijk}\hat{J}_k$$

and define $\hat{J}^2 = \hat{J}_i\hat{J}_i$.

   (a) Show that $\hat{J}^2$ and any one $\hat{J}_i$ can be measured simultaneously.

   (b) Suppose that $|\,\psi\rangle$ is a normalised eigenstate of $\hat{J}_3$ with eigenvalue $m\hbar$ for some constant m. Defining $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$, show that the two states $\hat{J}_\pm\,|\,\psi\rangle$ (assuming they are non-vanishing) are also eigenstates of $\hat{J}_3$, and find their eigenvalues.

   (c) Suppose the state $|\,\psi\rangle$ in part (b) is also an eigenstate of $\hat{J}^2$ and $\hat{J}_+\,|\,\psi\rangle = 0$. Find the eigenvalue of $\hat{J}^2$ in terms of m.

5. Consider a one-dimensional system with potential

$$V(x) = \begin{cases} 0 & , \quad x \leq 0 \\ U(x) & , \quad 0 < x < 3L \\ 0 & , \quad x \geq 3L \end{cases}$$

where $U(x)$ is some strictly positive function for all $x \in (0, 3L)$.

(a) Write the WKB wavefunction in the barrier region $0 < x < 3L$ for particles of mass m with energy $E > 0$, assuming $U(x) > E$ everywhere in this region.

(b) Suppose a beam of particles of mass m and energy $E > 0$ is incident from $x = -\infty$. Write the wavefunction in the regions $x < 0$ and $x > 3L$, and state the matching conditions to the wavefunction in the barrier region at $x = 0$ and at $x = 3L$.

(c) Give an expression in terms of $U(x)$, E, m, L and $\hbar$ which quantifies the size of the barrier. How can the WKB wavefunction in the barrier region be further simplified in the case where this quantity is large? (Note that the incoming particles are from $x = -\infty$ only.)

(d) Use the above results to find an order of magnitude estimate of the transmission coefficient, assuming a large barrier, for the specific case

$$U(x) = \begin{cases} V_0 & , \quad 0 < x < L \\ V_1 & , \quad L \leq x \leq 2L \\ V_2 & , \quad 2L < x < 3L \end{cases}$$

where V_0, V_1 and V_2 are positive constants.

6. The Hamiltonian for a one-dimensional simple harmonic oscillator is

$$\hat{H}_{SHO} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

and the normalised energy eigenstates are

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle$$

with eigenvalues $E_n = (n + 1/2)\hbar\omega$ for non-negative integer n, where $\hat{a}\,|0\rangle = 0$ and

$$\hat{a} = \sqrt{\frac{1}{2m\hbar\omega}} \left(m\omega\hat{x} + i\hat{p} \right) \; .$$

(a) If an additional term $\hat{H}' = \lambda\hat{p}^4$ with real parameter λ (assumed to be small) is added to the Hamiltonian, show that the first order perturbation of the energies $E_n \to E_n + \lambda E_n^1$ is given by

$$E_n^1 = \langle n |\, \hat{p}^4 \,| n \rangle.$$

(b) Calculate E_n^1. You may use the following results

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \; , \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle.$$

SECTION B

7. A Quantum Mechanical system has an observable $\hat{A}$ whose measurement can yield the values -1, 0 and 1. Its Hamiltonian $\hat{H}$ acts on the corresponding eigenstates of $\hat{A}$ as follows

$$\hat{H}\,|-1\rangle = |-1\rangle - i\,|1\rangle$$
$$\hat{H}\,|0\rangle = |0\rangle$$
$$\hat{H}\,|1\rangle = i\,|-1\rangle + |1\rangle\,.$$

(a) Find the matrix form of $\hat{H}$ and the observable $\hat{A}$ in the basis $\{|-1\rangle, |0\rangle, |1\rangle\}$.

(b) Find the eigenvalues and eigenstates of $\hat{H}$ expressed as a linear combination of $\{|-1\rangle, |0\rangle, |1\rangle\}$.

(c) At $t = 0$ a measurement of $\hat{A}$ yields its minimum possible value. Write down the state of the system for any $t > 0$.

(d) For the state that you found in the previous question, calculate the corresponding probabilities of all the possible values in a measurement of $\hat{A}$. Your answers should depend on time t.

(e) For the same state, find the expectation value $\langle \hat{A} \rangle$ as a function of time.

8. Consider a particle of mass m in the potential of a simple harmonic oscillator of characteristic frequency ω.

   (a) Write the Hamiltonian $\hat{H}$ of the system in terms of the position $\hat{x}$ and momentum $\hat{p}$ operators.

   (b) Define the annihilation operator

   $$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}\,(m\omega\,\hat{x} + i\,\hat{p})\ .$$

   Express $\hat{H}$ in terms of the annihilation and its adjoint (the creation operator).

   (c) The ground state $|0\rangle$ is such that it is annihilated by $\hat{a}$. Use this fact to find an explicit expression for the wavefunction of the ground state in position space i.e. write down $\psi_0(x) = \langle\, x\,|\,0\,\rangle$.
   Hint 1: The most general solution of the differential equation $f'(x) = 2\,\alpha\,x\,f(x)$ is $f(x) = c\,e^{\alpha\,x^2}$ for any constant c.
   Hint 2: You will need the Gaussian integral $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

   (d) Repeat part (c) for the wavefunction in momentum space i.e. write down $\tilde{\psi}_0(p) = \langle\, p\,|\,0\,\rangle$.

   (e) The system is in the ground state $|\,0\rangle$. The characteristic frequency of the potential suddenly changes from ω to ω'. The change happens so fast that the wavefunction of the state right after the change is the same. Calculate the probability for an experiment to find the particle in the new ground state of the system.

9. (a) The energy eigenvalues of the one-dimensional quantum system with Hamiltonian

$$\hat{H} = \frac{1}{2M}\hat{p}^2 + \frac{1}{2}M\omega^2\hat{x}^2$$

are $E_n = (n + 1/2)\hbar\omega$ for all non-negative integers n.

Briefly explain why you can immediately deduce that the energy eigenvalues of the three-dimensional quantum system with Hamiltonian

$$\hat{H} = \sum_{i=1}^{3}\left(\frac{1}{2M}\hat{p}_i^2 + \frac{1}{2}M\omega_i^2\hat{x}_i^2\right) \text{ are } E_{\boldsymbol{n}} = \sum_{i=1}^{3}\left(n_i + \frac{1}{2}\right)\hbar\omega_i$$

for all $\boldsymbol{n} = (n_1, n_2, n_3)$ with non-negative integer components.

What are the degeneracies of the states with energy $(N + 3/2)\hbar\omega$ for $N = 0$, $N = 1$ and $N = 2$ in the case where $\omega_1 = \omega_2 = \omega_3 = \omega$?

(b) For a particle of mass M in a central potential $V(r)$ in three dimensions, the time-independent Schrödinger equation can be solved using spherical polar coordinates in terms of a *radial wavefunction* $u(r)$ where

$$\psi(\boldsymbol{r}) = \frac{1}{r}u(r)Y_{lm}(\theta, \phi)$$

$$-\frac{\hbar^2}{2M}\frac{d^2}{dr^2}u + V_{eff}(r)u(r) = Eu(r)$$

$$V_{eff}(r) = V(r) + \frac{\hbar^2}{2M}\frac{l(l+1)}{r^2}.$$

i. The spherical harmonics $Y_{lm}(\theta, \phi)$ are simultaneous eigenstates of L^2 and L_z. State what the eigenvalues are and what values l and m can take.

ii. Consider now the case where

$$V = \frac{1}{2}M\omega^2 r^2$$

for some real constant ω. By approximating the radial equation for large r show that the radial wavefunction behaves like $\exp(-\rho^2/2)$ where ρ/r is a constant which you should determine.

iii. Writing the radial wavefunction in the form

$$u(r) = p(\rho)e^{-\rho^2/2}\ ,$$

derive the differential equation which $p(\rho)$ must satisfy.

Assuming $p(\rho)$ is a finite power series in ρ, determine the lowest power of ρ in $p(\rho)$, and from the assumed existence of a highest power, deduce that the allowed energy levels must be of the form $(N+3/2)\hbar\omega$ for non-negative integer N.

iv. Since the radial equation is invariant under the discrete transformation $\rho \to -\rho$, the radial wavefunction will be either an odd or an even function of ρ. Use this fact, with your results above to check that the degeneracies of states for $N = 0$, $N = 1$ and $N = 2$ agree with the values derived in part (a).

10. (a) Consider a particle of mass m in a one-dimensional potential $V(x)$. Write the wavefunction as

$$\psi(x) = A(x)e^{i\phi(x)}$$

where $A(x)$ and $\phi(x)$ are real functions. Use the time independent Schrödinger equation to find $A(x)$ in term of $\phi(x)$, and in a region where $E > V(x)$, find an integral expression for $\phi(x)$ in terms of E and $V(x)$ in the WKB approximation where we can assume that $A(x)$ varies slowly.

(b) Consider a one-dimensional system with potential

$$V = \begin{cases} \infty & , \quad x \leq -L \\ \frac{1}{2}m\omega^2 x^2 & , \quad -L < x < L \\ \infty & , \quad x \geq L \end{cases}$$

for some constant $\omega > 0$.

Write down the WKB approximation of the wavefunction when $E > \frac{1}{2}m\omega^2 L^2$ in the region $-L < x < L$ and state what the boundary conditions are. Use this to derive an implicit expression determining the quantised energies. (You do not have to solve explicitly for E.)

(c) Find explicit expressions for the approximate energy levels in the cases

- $E \gg \frac{1}{2}m\omega^2 L^2$
- $E \approx \frac{1}{2}m\omega^2 L^2$

Hint: For $-1 \leq u \leq 1$,

$$\int_0^u \sqrt{1-q^2}\mathrm{d}q = \frac{1}{2}\sin^{-1} u + \frac{1}{2}u\sqrt{1-u^2}.$$