EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH3141-WE01

Title:
Operations Research III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

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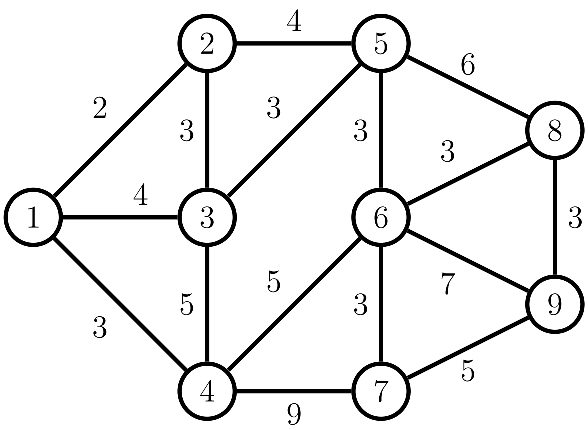
SECTION A

1. (a) Solve the following linear program using the simplex method:

$$\begin{array}{rrcl} \max & -2x_1 - 4x_2 + x_3 & & \\ \text{subject to} & x_1 - 3x_2 + x_3 & \leq & 2 \\ & 3x_1 - 5x_2 + x_3 & \leq & 1 \\ & x_1 & \geq & 0 \\ & x_2 & \geq & 0 \\ & x_3 & \geq & 0 \end{array}$$

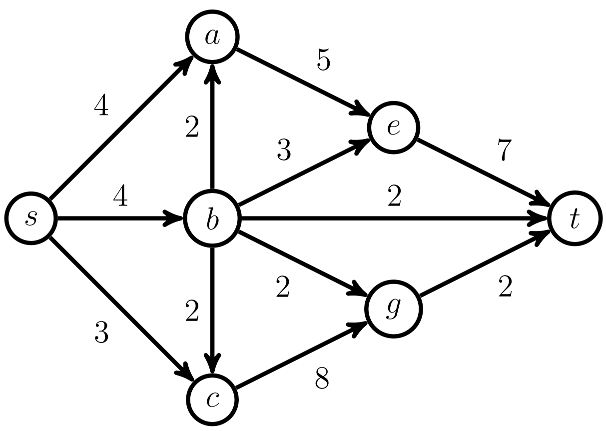
(b) State the complementary slackness condition. From the solution in your final table, verify that this condition is satisfied. In one sentence, explain why this condition must be satisfied for your solution.

2. Consider the following network:



The numbers on the arcs denote distances between nodes. Apply Dijkstra's algorithm to find the shortest distance from node 1 to node 9. Identify the corresponding shortest paths from node 1 to node 9.

3. Consider the following flow network, with capacities as labelled:



(a) Apply the Ford-Fulkerson algorithm to find the maximum flow from s to t.

(b) Find a cut separating s and t with minimal capacity.

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4. Use dynamic programming to solve the following non-linear programming problem (not restricted to the integers)

$$\text{maximize } Z = 36x_1 + 9x_1^2 - 6x_1^3 + 36x_2 - 3x_2^3$$

subject to $x_1 + x_2 \leq 3$ and $x_1 \geq 0$ and $x_2 \geq 0$.

5.

6. (a) Let $(X_n)_{n\geq 0}$ be a time homogeneous Markov chain on a finite state space S. Prove the Chapman Kolmogorov equations, that is

$$\mathbb{P}[X_{n+m} = j | X_0 = i] = \sum_{k\in S} \mathbb{P}[X_n = k | X_0 = i]\mathbb{P}[X_m = j | X_0 = k]$$

for $i, j \in S$.

(b) Consider a Markov chain with state space $\{1, 2, 3\}$ with transition matrix given by

$$\begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.3 & 0.3 \\ 0 & 1 & 0 \end{pmatrix}$$

What is the probability of reaching state 1 before state 3 when starting from state 2?

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SECTION B

7. Consider the following linear programming problem:

$$\begin{aligned} \max\ & \alpha x_1 + 4x_2 + 3x_3 + 5x_4 \\ \text{subject to}\ & 2x_1 + x_2 + 3x_3 + 2x_4 + s_1 = \beta \\ & 3x_1 + x_2 + x_3 + 2x_4 + s_2 = 2 \\ & 2x_1 + x_2 + 3x_3 + x_4 + s_3 = 1 \end{aligned}$$

and subject to all $x_i \geq 0$ and $s_i \geq 0$. In the above, α and β are fixed parameters in $\mathbb{R}$.

(a) For $\alpha = 2$ and $\beta = 5$, the final simplex table for this problem is

T_*	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
z	7	0	7	0	0	1	3	5
s_1	-1	0	2	0	1	-1	0	3
x_4	1	0	-2	1	0	1	-1	1
x_2	1	1	5	0	0	-1	2	0

Use post-optimal analysis to solve the problem for $\alpha = 4$ and $\beta = 1$.

(b) Consider $\alpha = 2$ and $\beta = 5$ (as for T_* above). A new variable is added to the problem (indicated in bold):

$$\begin{aligned} \max\ & 2x_1 + 4x_2 + 3x_3 + 5x_4 + \boldsymbol{6x_5} \\ \text{subject to}\ & 2x_1 + x_2 + 3x_3 + 2x_4 + \boldsymbol{2x_5} + s_1 = 5 \\ & 3x_1 + x_2 + x_3 + 2x_4 + \boldsymbol{x_5} + s_2 = 2 \\ & 2x_1 + x_2 + 3x_3 + x_4 + \boldsymbol{x_5} + s_3 = 1 \end{aligned}$$

and again subject to all $x_i \geq 0$ and $s_i \geq 0$. Use post-optimal analysis to solve this problem.

8. Consider the transportation problem with costs, supplies, and demands respectively given by:

$$[c_{ij}] = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 3 & 1 & 3 & 3 \\ 1 & 2 & 3 & 2 & 1 \end{bmatrix} \qquad [a_i] = \begin{bmatrix} 3 & 7 & 5 \end{bmatrix} \qquad [b_j] = \begin{bmatrix} 4 & 8 & 1 & 1 & 1 \end{bmatrix}$$

(a) Find the optimal transportation scheme.

(b) Consider an extended version of the above problem, where a new source and destination are added (with changes indicated in bold):

$$[c_{ij}] = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 & \mathbf{1} \\ 3 & 3 & 1 & 3 & 3 & \mathbf{1} \\ 1 & 2 & 3 & 2 & 1 & \mathbf{1} \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & \mathbf{5} & \mathbf{5} & \mathbf{4} \end{bmatrix} \quad [a_i] = \begin{bmatrix} 3 & 7 & 5 & \mathbf{3} \end{bmatrix} \quad [b_j] = \begin{bmatrix} 4 & 8 & 1 & 1 & 1 & \mathbf{3} \end{bmatrix}$$

Explain how you can use the optimal solution of the initial problem to more quickly obtain an optimal solution of this extended problem.

(c) Find the optimal transportation scheme for the extended problem.

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9. (a) An optimization problem can be divided into N stages with possible states $s \in \mathcal{S}_n$ and actions $x \in \mathcal{A}_n$ at stage $n \leq N$. Let $\mathcal{S}_{N+1}$ denote the final set of states. The probability of going to state $j \in \mathcal{S}_{n+1}$ from state i after action x in stage n is $P_n(i, j; x)$. Write down the backwards induction algorithm for stochastic dynamic programming for maximizing the probability of the event $F \subset \mathcal{S}_{N+1}$.
Define carefully any notation you introduce.

A poor but very friendly statistician has established a system for winning a popular two player game. Her colleagues do not believe her system and decide to play three separate games against her. For each game, she has the opportunity to place an even bet, meaning that after each game she stands to win or lose the same amount of money that she has bet on that game. The amount she bets in each game can be *any* quantity of her choice between zero and the amount of money she has left after the bets of the preceding games. The statistician believes that her system will give her a probability of $\frac{2}{3}$ of winning a given game and thus winning the amount bet and a probability of $\frac{1}{3}$ of losing a given game and thus losing the amount bet. She begins with £3 and her goal is to have exactly £5 at the end (since these are games with colleagues, she does not want to end up with more than £5). Therefore, she wants to find the optimal betting policy that maximizes the probability that she will have exactly £5 after the three games.

Let n denote the n^{th} play of the game, x_n denote the number of pounds bet at stage n, s_n denote the number of pounds she has available at stage n. Let $f_n(s_n; x_n)$ denote the probability of finishing with exactly £5 at the end of the three games given that she started the game n with £s_n and betted £x_n. Let

$$f_n^*(s_n) = \max_{x_n \in \{0,\ldots,s_n\}} f_n(s_n; x_n).$$

(b) Using part (a), find the optimal betting policy that maximizes the probability that the statistician will have exactly £5 after three games.

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10. A local golf course, which charges £20 per round, must decide upon a course maintenance policy. The state of the course over any given week can be 1 (pristine), 2 (acceptable), or 3 (shabby). A pristine course attracts 390 paying rounds per week; for acceptable the figure is 210 rounds, and for shabby just 90.

At the start of each week, the management will call upon the services of either Fred Shanks the greenkeeper, who charges £1200 per week, or Geoff Divito the local handyman, who charges £600 per week. This decision is made solely on the basis of the state of the course in the previous week.

The efforts of Fred and Geoff cause the state of the course for the week in which they are employed to be updated in a Markov fashion depending on the state of the course in the previous week, according to the transition probabilities

Fred:

Last week \ This week	1	2	3
1	5/6	1/6	0
2	0	1	0
3	0	1/4	3/4

Geoff:

Last week \ This week	1	2	3
1	1/2	1/4	1/4
2	0	1/4	3/4
3	0	0	1

If the state of the course for the previous week was shabby, the management may decide instead to take drastic measures and close the course for the coming week. If this happens, they will employ Fred at his usual rate, and, uninterrupted by golfers, he will restore the course to pristine condition by the end of the week, but the course takes no revenue for that week.

The goal is to find a policy to maximize the expected discounted total return, where the (weekly) discount rate is $\alpha = 4/5$.

(a) Formulate the problem as a Markov decision process by identifying the states, decisions, the one-step transition probability matrices corresponding to these decisions, and the expected immediate returns $r(i;k)$ when in state i decision k is taken.

The management is currently implementing Policy A_1, in which Fred is employed every week, whatever the state of the course.

(b) Starting with Policy A_1, perform *two* complete iterations of the policy improvement algorithm to find an improved policy.

You do not need to show that this policy is optimal.

Hint: The features of the problem are such that you can re-use several previous calculations for the second iteration.

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