

EXAMINATION PAPER

Examination Session: May

2018

Year:

Exam Code:

MATH3251-WE01

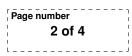
Title:

Stochastic Processes III

Time Allowed:	3 hours			
Additional Material provided:	None			
Materials Permitted:	None			
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.		
Visiting Students may use dictionaries: No				

Instructions to Candidates:	Credit will be given for: the best FOUR answers from Section A and the best THREE answers from Sec Questions in Section B carry TWICE as in Section A.	ction B.	irks as those
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Revision:



SECTION A

- 1. Let $(X_n)_{n\geq 0}$, $X_0 = 1$, be a branching process with Binomial Bin(m, p) offspring distribution, and let $(Y_n)_{n\geq 0}$, $Y_0 = 1$, be a branching process with Poisson $Poi(\lambda)$ offspring distribution.
 - (a) Carefully describe all values of the parameters for which the processes X_n and Y_n are subcritical and for which they are supercritical.
 - (b) Let mp = λ. Which of the two processes is more likely to survive forever? Justify your answer.
 [Hint: You may use without proof the following inequality: 1 + x ≤ e^x for all real x.]

In your answer you should carefully define branching processes and extinction probabilities, and give a clear statement of any result you use.

2. Let $(X_k)_{k\geq 1}$ be a sequence of independent identically distributed random variables with $\mathsf{P}(X_1 = \pm 1) = 1/2$. Let $(S_n)_{n\geq 0}$ be the simple symmetric random walk generated by $(X_k)_{k\geq 1}$, namely, $S_0 = 0$ and $S_n = \sum_{k=1}^n X_k$ for n > 0. Define

$$H \stackrel{\mathsf{def}}{=} \inf\{n \ge 0 : S_n = 1\}.$$

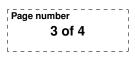
- (a) Carefully show that S_n is a martingale and that H is a stopping time for $(S_n)_{n\geq 0}$.
- (b) Show that none of the three versions of the optional sampling theorem can be applied to S_H . In each case carefully show which assumptions of the corresponding theorem are violated.

In your answer you should carefully define martingales and stopping times, and give a clear statement of any result you use.

[**Hint:** You can use without proof the following result: $\mathsf{E}s^H = (1 - \sqrt{1 - s^2})/s$.]

- 3. Let X and Y be discrete random variables with distributions $\mathsf{P}(X = x) = p_x$ and $\mathsf{P}(Y = y) = q_y$.
 - (a) Define the maximal coupling of X and Y; check that it has the correct marginal distributions.
 - (b) In addition, suppose that X is Bernoulli distributed with parameter r and Y is Poisson distributed with parameter r. Derive the maximal coupling of X and Y; deduce the total variation distance between the distributions of X and Y.
- 4. (a) Carefully define the Poisson process $(N(t))_{t\geq 0}$ with intensity $\lambda > 0$, and state its main properties.
 - (b) For fixed t > 0, write the formula for the distribution of N(t), i.e., for the values of probabilities P(N(t) = k) with integer $k \ge 0$. Prove your formula for the case k = 0.
 - (c) Let fixed t_1 , t_2 and t_3 satisfy $0 < t_1 < t_2 < t_3$. Find the conditional probability

$$\mathsf{P}(N(t_2) - N(t_1) = 2 \mid N(t_3) = 3).$$





- 5. Let $(X_k)_{k\geq 1}$ be independent random variables, where $X_k \sim \mathsf{Exp}(\lambda_k)$ with $\lambda_k \in (0, \infty)$ for all k. Denote $T = \sum_{k\geq 1} X_k$.
 - (a) If $\sum_{k>1} 1/\lambda_k < \infty$, carefully show that $\mathsf{P}(T < \infty) = 1$.
 - (b) If $\sum_{k\geq 1} 1/\lambda_k = \infty$, carefully show that $\mathsf{P}(T = \infty) = 1$.

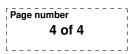
In your answer you should carefully state any result you use.

6. Let $X_1(t)$ and $X_2(t)$ be continuous-time Markov chains on $\{0, 1\}$ with common generator

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}, \qquad \alpha, \beta > 0.$$

Assuming that the processes $X_1(t)$ and $X_2(t)$ are independent (i.e., $X_1(u)$ and $X_2(v)$ are independent for all non-negative u and v), let $X(t) = X_1(t) + X_2(t)$.

- (a) Show that X(t) is a continuous-time Markov chain and find its generator.
- (b) Write down the forward Kolmogorov equations for X(t).
- (c) Find the stationary distribution of X(t).



SECTION B

- 7. Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be two branching processes.
 - (a) If $X_0 \leq Y_0$ and both processes have the same offspring distribution $(\mathbf{p}_m)_{m\geq 0}$, show that X_n is stochastically dominated by Y_n (i.e., $X_n \preccurlyeq Y_n$) for all $n \ge 0$. Deduce that the Y-process is more likely to survive than the X-process.
 - (b) Suppose now $X_0 = Y_0 = 1$, but the offspring distribution of the X-process is stochastically smaller than the offspring distribution of the Y-process, equivalently, $X_1 \preccurlyeq Y_1$. Show that X_n is stochastically dominated by Y_n for all $n \ge 0$. Deduce that the Y-process is more likely to survive than the X-process.
- 8. A standard fair die is tossed repeatedly. Let T be the number of tosses until the sequence 1-2-3-3-2-1 is observed for the first time. Use the appropriate optional stopping theorem to find the expectation $\mathsf{E}T$.

In your answer you should clearly state and carefully apply any result you use.

9. Let $X(t), t \ge 0$, be a continuous-time Markov chain on the state space $\{0, 1, 2\}$ whose generator (*Q*-matrix) is

$$Q = \begin{pmatrix} -3 & 1 & 2\\ 2 & -2 & 0\\ 1 & 1 & -2 \end{pmatrix}$$

- (a) Write down the backward Kolmogorov equations for the transition probabilities $p_{ij}(t), i, j \in \{0, 1, 2\}.$
- (b) Show that

$$P(t) = \exp\{tQ\} = \sum_{k\geq 0} \frac{t^k}{k!} Q^k$$

is a unique solution to the equations you obtained in part (a).

(c) Define the resolvent $R(\lambda)$ for X(t) and find

$$p_{12}(t) = \mathsf{P}(X(t) = 2 \mid X(0) = 1).$$

- 10. (a) Carefully define a Brownian motion, state its Markov and strong Markov properties.
 - (b) Let $(B_t)_{t\geq 0}$ be a Brownian motion starting at the origin, $B_0 = 0$, and let T_a be the hitting time of a > 0, i.e., $T_a = \inf\{t \ge 0 : B_t = a\}$. Use the reflection principle to show that

$$\mathsf{P}(T_a \le t) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t} \, dx \, ,$$

and find the probability density function of T_a .

[Hint: You might wish to change the variables $x \mapsto s$ with $x = t^{1/2}a/s^{1/2}$.]

(c) Carefully show that for all $0 \le s < t$ we have

$$\mathsf{E}(B_t|B_r, 0 \le r \le s) = B_s, \qquad \mathsf{E}(B_t^2 - t|B_r, 0 \le r \le s) = B_s^2 - s,$$

ie., that B_t and $B_t^2 - t$ are martingales with respect to the natural filtration.