EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH3281-WE01

Title:
Topology III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

The following conventions hold in this paper

- $\mathbb{R}$ denotes the space of all real numbers with the Euclidean topology.
- $\mathbb{C}$ denotes the space of all complex numbers with the Euclidean topology.
- $\mathbb{R}^n$ denotes the real n-dimensional space with the Euclidean topology.

SECTION A

1. (a) State the definition of a Hausdorff space.

   (b) Let $X = \{1, 2, 3, 4, 5\}$. Give an example of a Hausdorff topology on X, and an example of a non-Hausdorff topology on X.

   (c) Assume that X is a Hausdorff space. Show that

   $$\Delta(X) = \{(x, x) \in X \times X \mid x \in X\}$$

   is a closed subset of $X \times X$.

2. Let X be a topological space and $A \subset X$.

   (a) State the definition of a limit point of A.

   (b) Show that if x is a limit point of A, then $x \in \overline{A}$, the closure of A.

   (c) Determine the limit points of

   $$A = \left\{\cos\left(\tfrac{1}{n}\right) \in \mathbb{R} \mid n \in \mathbb{Z} - \{0\}\right\}.$$

3. (a) State the definition of a path-connected space.

   (b) Let

   $$X = \left\{\left(\cos\left(\tfrac{1}{x}\right), \sin\left(\tfrac{1}{x}\right), x\right) \in \mathbb{R}^3 \mid x \in (0, 1)\right\}.$$

   Decide whether X is path-connected or not. Justify your statement.

4. (a) Suppose $\lambda\colon [0, 1] \to X$ and $\mu\colon [0, 1] \to X$ are two loops in the space X based at the point x_0. Define the product loop $\lambda * \mu\colon [0, 1] \to X$.

   (b) Now suppose λ, μ, κ and ν are four loops in X based at x_0. Prove that in general the loops $(\lambda * \mu) * (\kappa * \nu)$ and $\lambda * (\mu * (\kappa * \nu))$ are not the same. Prove, using a suitable diagram, that they represent however the same element of $\pi_1(X, x_0)$. (You do not have to give explicit formulas.)

5. (a) Let $S = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane, and $f : \mathbb{C} \to \mathbb{C}$ a continuous function. Let $W(f)$ be the winding number of the loop $f(S)$ about the origin: explain precisely what this means and what assumption must be made about f for it to be defined.

   (b) Say if each of the following statements are true or false in general, justifying your answers. In both statements, r is a non-zero complex number.

   (i) $W(rf) = W(f)$ where rf is the function $z \mapsto r \cdot f(z)$.

   (ii) $W(fr) = W(f)$ where fr is the function $z \mapsto f(rz)$.

6. (a) If A and B are finite simplicial complexes, state a formula relating the Euler characteristics of A, B, $A \cup B$ and $A \cap B$. State, without proof, the Euler characteristics of the sphere S^2, the circle S^1 and the closed 2-disc D^2. Use your formula to compute the Euler characteristic of the space Y which is given by removing three non-intersecting open discs from a sphere S^2.

   (b) A closed surface is constructed by taking 26 copies of the space Y and identifying pairs of boundary circles in such a way that the final space is connected and without boundary. How many different closed surfaces (up to homeomorphism) can be constructed in this way? Justify your answer, identifying the ones you think can be constructed in terms of the families of surfaces given in the Classification Theorem of Closed Surfaces considered in lectures, explaining any notation you use.

SECTION B

7. Let $X = \mathbb{R}^n \cup \{\infty\}$, that is, the set consisting of $\mathbb{R}^n$ and one extra element denoted ∞ (note that X is not a topological space yet). In particular, we have $\mathbb{R}^n \subset X$, but $\mathbb{R}^n \neq X$.

   (a) Let $U \subset \mathbb{R}^n$ be open and $C \subset \mathbb{R}^n$ compact. Show that $U \cap (X - C)$ is an open subset of $\mathbb{R}^n$.

   (b) Let $U \subset \mathbb{R}^n$ be open and $C \subset \mathbb{R}^n$ compact. Show that $U \cup (X - C) = X - D$ for a compact set $D \subset \mathbb{R}^n$.

   (c) Let
   $$\tau = \{U \subset \mathbb{R}^n \,|\, U \text{ open in } \mathbb{R}^n\} \cup \{X - C \,|\, C \subset \mathbb{R}^n \text{ compact}\}$$
   which is a subset of the power set of X, as $\mathbb{R}^n \subset X$.

      (i) Show that τ is a topology.

      (ii) Show that X with this topology is compact.

      (iii) For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let $|x|^2 = x_1^2 + \cdots + x_n^2$. Show that the function $f \colon X \to \mathbb{R}^n \times \mathbb{R}$ given by
      $$f(x) = \begin{cases} \left(\frac{2}{1+|x|^2} \cdot x, 1 - \frac{2}{1+|x|^2}\right) & \text{if } x \in \mathbb{R}^n \\ (0, 1) & \text{if } x = \infty \end{cases}$$
      is a homeomorphism onto its image.
      (Remark: We write $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ to emphasize that we only need two coordinates, an $\mathbb{R}^n$-coordinate and an $\mathbb{R}$-coordinate. It is possible to do this with $(n+1)$ real coordinates, which only means you have to write more.)

8. Let $U(n) = \{A \in GL_n(\mathbb{C}) \mid AA^* = I\}$, where $A^* = (a^*_{ij})$ is the matrix whose entries satisfy $a^*_{ij} = \bar{a}_{ji}$, where $A = (a_{ij})$, and $\bar{a}$ is the complex conjugate of $a \in \mathbb{C}$.

   (a) Show that for $A \in U(n-1)$ the matrix
   $$i(A) = \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$
   satisfies $i(A) \in U(n)$.

   (b) Show that $U(n-1)$ acts on $U(n)$ by
   $$A \cdot B = Bi(A^*)$$
   where $A \in U(n-1)$, $B \in U(n)$, and we use matrix multiplication on the right hand side.

   (c) Show that $f\colon U(n) \to S^{2n-1}$ given by
   $$f(B) = Be_n,$$
   where $e_n = (0, \ldots, 0, 1) \in \mathbb{C}^n$, induces a map $\bar{f}\colon U(n)/U(n-1) \to S^{2n-1}$ which is a homeomorphism. Here
   $$S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1\bar{z}_1 + \cdots + z_n\bar{z}_n = 1\}.$$

   (d) Show that $U(n)$ is connected for all $n \geq 1$.

   State any results from the lectures that you are using. You may use that $U(n)$ is compact.

9. (a) Give a net for the Klein bottle K as a quotient of a unit square. Triangulate your square so as to give a triangulation of K and use it to compute $\pi_1(K)$, briefly explaining any diagrams you use.

   (b) Using two copies of your triangulated square, pieced together appropriately, show that there is a triangulated orientable closed surface W and a map $f\colon W \to K$ with the property that for each n-simplex σ in K, there are precisely two n-simplices, τ_1, τ_2 in W such that $f(\tau_i) = \sigma$ for $i = 1, 2$.

   (c) Identify the surface W.

   (d) Use your triangulation of W to compute $\pi_1(W)$.

   (e) For each of your spaces, pick a point to be its basepoint and interpret the generators of your fundamental groups as loops based at this point. Use this to describe the homomorphism $f_*\colon \pi_1(W) \to \pi_1(K)$.

10. (a) Prove that $\chi(G)$, the Euler characteristic of a connected graph, is at most 1, with $\chi(G) = 1$ if and only if G is a tree.

    (b) Define $\chi(S)$, the Euler characteristic of a closed surface, in terms of a suitable graph drawn on it. Explain what has to be checked to make sure your definition is unambiguous, but you do not need to do the checking.

    (c) Sketch the proof of the theorem that the Euler characteristic of a closed surface is at most 2, and is equal to 2 if and only if S is a sphere. You should explain the main stages and structure of the proof, but you can assume the concepts and results from lectures that you need, so long as you state them correctly.

    (d) State how the Euler characteristic can be defined for a (finite) simplicial complex.

    (e) Let G and H be two 1 dimensional simplicial complexes. Explain how their cartesian product $G \times H$ can be given the structure of a simplicial complex, and use your construction to describe $\chi(G \times H)$ in terms of $\chi(G)$ and $\chi(H)$.