Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH3391-WE01

Title: Quantum Information III

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	No	Models Permitted: Use of electronic calculators is forbidden.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **FOUR** answers from Section A and the best **THREE** answers from Section B. Questions in Section B carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. Consider a single qubit Hilbert space with states represented by column vectors. Two operators are represented by matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; , \quad B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} .$$

   (a) Briefly explain why A and B are observables but show that they cannot be measured simultaneously.

   (b) What are the possible values, and the corresponding probabilities, of a measurement of A on the state $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

   (c) Suppose the measurement of A gave the highest possible value. What are the possible outcomes and probabilities if A is measured again, or if instead B is measured? Finally, what happens if A is then measured assuming the measurement of B gave the lowest possible value?

2. Consider a two-qubit system in a state

$$|\Psi\rangle = \cos\theta \, |00\rangle + e^{i\phi} \sin\theta \, |11\rangle$$

for some real constants θ and ϕ.

   (a) Calculate the density operator $\hat{\rho}$.

   (b) If the system is a bipartite system (consisting of two one-qubit subsystems) calculate the reduced density operator $\hat{\rho}_A$ in one subsystem.

   (c) Calculate $\mathrm{Tr}(\hat{\rho}_A^2)$ and comment on its interpretation. State (without proof) what can happen to $\mathrm{Tr}(\hat{\rho}_A^2)$ under local operations in either subsystem.

3. The density matrix ρ for a single qubit can be described in terms of the Bloch sphere as

$$\rho = \frac{1}{2}\left(I + \mathbf{r}\cdot\sigma\right)$$

where the Bloch vector $\mathbf{r}$ is a position vector in three dimensions, I is the identity matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \; , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

   (a) What are the allowed values of $r = |\mathbf{r}|$ and how are pure states distinguished from mixed states in the Bloch sphere picture?

   (b) Suppose the system has Hamiltonian $H = \alpha\sigma_3$ for some constant α. Calculate the density matrix at time $t > 0$ in terms of the density matrix at time $t = 0$ and state the geometric interpretation (in the Bloch sphere picture) of this time-evolution.

4. Consider a classical binary function $f(x)$, so $f(x)$ is either 0 or 1 for each input x.

   (a) Describe the circuit model, where $f(x)$ is constructed in terms of a set of elementary operations.

   (b) Take the input x to have n bits. Determine the number of distinct functions $f(x)$.

   (c) In reversible computation, we take the elementary operations to be NOT, controlled-not (CNOT), and controlled-controlled not (CCNOT). Give the number of possible circuits you can construct acting on n bits with m of these operations.

   (d) Hence argue that we will need circuits with exponentially many (in n) operations to construct all the functions $f(x)$ of an n bit input.

5. Consider a two-qubit system consisting of qubits q_1, q_0. We act on it with a unitary operation constructed by first acting with NOT on q_1, then the Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ on q_0, then a controlled-not with control bit q_1 and target bit q_0, and finally another Hadamard on q_0.

   (a) Draw the corresponding quantum circuit.

   (b) Determine the action on computational basis states.

   (c) Hence give the matrix representation of this unitary operation.

6. We wish to devise a quantum error correcting code to encode a single logical qubit in n physical qubits so as to protect from single bit flip errors. Explain why we need $n + 1$ orthogonal two-dimensional subspaces in the physical Hilbert space. Hence show the minimum possible number of physical qubits is 3. Similarly determine the minimum possible number of physical qubits to enable recovery from arbitrary single qubit errors.

SECTION B

7. (a) Suppose a system has 4 measurable quantities, Q, R, S and T. Briefly state what *local realism* says about measurement of one or more of these quantities.

(b) Assuming local realism, and that a measurement of any one of Q, R, S and T gives a value either $+1$ or -1, what are the possible values of a measurement of $(QS + RS + QT - RT)$? Hence derive the best possible upper and lower bounds on the average value of $(QS+RS+QT-RT)$ if measured for a number of such systems (each of which may be in any allowed state.)

(c) Now consider a two-qubit quantum system and represent the states as 4-component column vectors with

$$|0\rangle\otimes|0\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle\otimes|1\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle\otimes|0\rangle \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1\rangle\otimes|1\rangle \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Also define

$$Q = \sigma_1 \otimes I \ , \quad R = \sigma_3 \otimes I \ , \quad S = \frac{-1}{\sqrt{2}} I \otimes (\sigma_1 + \sigma_3) \ , \quad T = \frac{-1}{\sqrt{2}} I \otimes (\sigma_1 - \sigma_3).$$

State which pairs of these observables cannot be simultaneously measured.
Write down the 4×4 matrix representation of Q, R, S and T, noting that

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(d) Now calculate the expectation values $\langle QS \rangle$, $\langle RS \rangle$, $\langle QT \rangle$ and $\langle RT \rangle$ in the state

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}} \big(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \big).$$

(e) Describe how your results above can be used in a bipartite system to demonstrate that the postulates of local realism are not satisfied. In particular, state which measurements should be performed and state why it is important to consider two separated subsystems.

8. (a) State the no-cloning theorem in quantum mechanics.

(b) Suppose there is some process in quantum mechanics which clones two specific states $|\phi\rangle$ and $|\psi\rangle$ in the sense that for some fixed state $|\Omega\rangle$ the process maps (with probability 1)

$$\begin{aligned} |\phi\rangle \otimes |\Omega\rangle &\rightarrow |\phi\rangle \otimes |\phi\rangle \\ |\psi\rangle \otimes |\Omega\rangle &\rightarrow |\psi\rangle \otimes |\psi\rangle \end{aligned}$$

Using the assumption of linearity, show that this process cannot clone a generic state.

(c) Briefly describe a method that clones a state provided it is one of two known orthogonal states $|\phi\rangle$ and $|\psi\rangle$.

(d) Now suppose that Alice has a qubit in an unknown state $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$. She also shares a Bell state with Bob so that the full bipartite system is initially in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\,|\psi\rangle \otimes \Big(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle \Big) .$$

Calculate the state of the system after Alice performs a unitary transformation on the first 2 qubits which leaves $|0\rangle \otimes |0\rangle$ and $|0\rangle \otimes |1\rangle$ unchanged while exchanging $|1\rangle \otimes |0\rangle \leftrightarrow |1\rangle \otimes |1\rangle$, and then performs a unitary transformation on the first qubit only, mapping

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle \Big) \ \text{ and } \ |1\rangle \rightarrow \frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle \Big) .$$

(e) Alice now measures both her qubits using the observable $|1\rangle\langle 1|$. What is the probability that Bob now has the state $|\psi\rangle$? Suppose that Alice got result 0 for both measurements. Describe explicitly what Alice and Bob must do using LOCC only to ensure that Bob has the state $|\psi\rangle$.

(f) Why is the above process consistent with the no-cloning theorem?

Comment on whether Alice and Bob could use this process to communicate by Alice choosing the state $|\psi\rangle$. In particular, does the use of entanglement in this process allow for more efficient classical communication, or are there advantages in terms of secrecy assuming the classical communications are not secure.

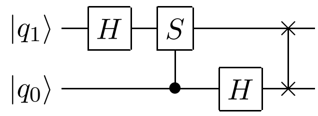
9. Consider the Quantum Fourier Transform, defined as the unitary operator U_{FT} on an n qubit Hilbert space whose action on basis states $|x\rangle$, $x = 0, \dots 2^n - 1$ is

$$U_{FT}|x\rangle = \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n} |y\rangle.$$

Consider the case $n = 2$.

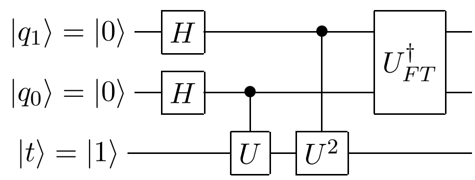
(a) Evaluate the action of U_{FT} on the computational basis.

(b) Show that this is reproduced by the circuit



where $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, and the last operation swaps the two qubits.

(c) **Phase estimation:** Suppose we have a unitary U, which we know is either the identity, S, or $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and we want to determine which. Show that the circuit



will allow us to determine the value of U by measuring the output values of q_1, q_0 in the computational basis.

10. (a) Show that the operator

$$R_\chi = I - 2|\chi\rangle\langle\chi|,$$

where I is the identity, generates a reflection in the plane orthogonal to the vector $|\chi\rangle$ in the Hilbert space. Show that this is a unitary operator for any vector $|\chi\rangle$.

(b) Suppose that we have R_χ, and we want to determine $|\chi\rangle$. Starting with a trial vector $|\psi\rangle$ whose overlap with $|\chi\rangle$ is small, show how we can obtain a vector with larger overlap with $|\chi\rangle$, by acting with R_χ and $-R_\psi$.

How large would the initial overlap have to be for this operation not to improve it?

(c) Suppose we obtain a quantum system in the state $|\chi\rangle$. Does this operation allow us to determine the state?

(d) Suppose we know the state $|\chi\rangle$ is one of the computational basis states. Give a suitable choice of trial vector $|\psi\rangle$, and determine the number of applications of R_χ and $-R_\psi$ required to bring the state of the system as close as possible to $|\chi\rangle$.