Durham University

EXAMINATION PAPER

Examination Session:	Year:	Exam Code:
May	2018	MATH4041-WE01

Title:
Partial Differential Equations IV

Time Allowed:	3 hours	
Additional Material provided:	None	
Materials Permitted:	None	
Calculators Permitted:	Yes	Models Permitted: Casio fx-83 GTPLUS or Casio fx-85 GTPLUS.
Visiting Students may use dictionaries: No		

Instructions to Candidates:	Credit will be given for: the best **TWO** answers from Section A, the best **THREE** answers from Section B, **AND** the answer to the question in Section C. Questions in Section B and C carry **TWICE** as many marks as those in Section A.

Revision:	

ED01/2018

SECTION A

1. Consider the conservation law

$$\begin{cases} u_t + uu_x = 0 & \text{for } (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = e^{-(x-1)^2} & \text{for } x \in \mathbb{R}. \end{cases} \tag{1}$$

   (a) Find the largest value of $T \in \mathbb{R}$ for which this conservation law has a classical solution $u : \mathbb{R} \times [0,T) \to \mathbb{R}$.

   (b) Give a sketch of the characteristics of (1) up until time T.

   (c) Find an implicit equation for u that does not contain partial derivatives.

2. Consider the conservation law

$$\begin{cases} u_t + (f(u))_x = 0 & \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{2}$$

   (a) Define the Rankine-Hugoniot condition for a shock.

   (b) Define the Lax entropy condition for a shock.

   (c) Let

$$g(x) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x > 0, \end{cases} \qquad \text{and} \qquad f(u) = \frac{1}{4}u^4.$$

   Consider the functions

$$u_1(x,t) = \begin{cases} 1 & \text{for } x < t/2, \\ 0 & \text{for } x > t/2, \end{cases} \qquad \text{and} \qquad u_2(x,t) = \begin{cases} 1 & \text{for } x < t/4, \\ 0 & \text{for } x > t/4. \end{cases}$$

   Which one of these two functions is a weak solution of (2)?
   Is it also an admissible solution in the sense of the Lax entropy condition?

3. State the theorem of local existence for first-order quasilinear PDEs

$$\begin{cases} a_1(x,y,u)u_x + a_2(x,y,u)u_y = b(x,y,u) & \text{for } (x,y) \in \Omega, \\ u(x,y) = u_0(x,y) & \text{for } (x,y) \in \Gamma, \end{cases}$$

   in the following steps:

   (a) State the assumptions on the Cauchy data Γ and u_0.

   (b) State the assumptions on the coefficients a_1, a_2, b.

   (c) State what is means for the data to be noncharacteristic.

   (d) State the result of the theorem.

4. (a) Let $B_1(\mathbf{0}) \subset \mathbb{R}^2$ be the unit ball in two dimensions. Suppose that $u : \overline{B_1(\mathbf{0})} \to \mathbb{R}$ satisfies

$$\begin{aligned} -\Delta u &= g && \text{in } B_1(\mathbf{0}), \\ \nabla u \cdot \boldsymbol{n} &= 1 + \alpha && \text{on } \partial B_1(\mathbf{0}), \end{aligned}$$

where

$$g(\boldsymbol{x}) = \begin{cases} \frac{1}{|\boldsymbol{x}|} \sin(2\pi|\boldsymbol{x}|) & \text{if } \boldsymbol{x} \neq \mathbf{0}, \\ 2\pi & \text{if } \boldsymbol{x} = \mathbf{0}, \end{cases}$$

and where $\boldsymbol{n}$ is the outward-pointing unit normal vector field to $\partial B_1(\mathbf{0})$, and $\alpha \in \mathbb{R}$ is a constant. Find α.

(b) State the Poincaré inequality for functions $f \in C^1([a,b])$.

5. (a) State the strong maximum principle for harmonic functions.

(b) Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

and let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{aligned} -\Delta v &= f && \text{in } \Omega, \\ v &= h && \text{on } \partial\Omega, \end{aligned}$$

where $f, g, h : \overline{\Omega} \to \mathbb{R}$. Assume that $g \leq h$ and that there exists $\boldsymbol{x}_0 \in \partial\Omega$ such that $g(\boldsymbol{x}_0) < h(\boldsymbol{x}_0)$. Show that $u < v$ in Ω.

6. Let $\Omega \subset \mathbb{R}^2$ be open and bounded with smooth boundary. Let

$$V = \{\varphi \in C^1(\overline{\Omega}) \, : \, \varphi = 0 \text{ on } \partial\Omega\}.$$

Define $E : V \to \mathbb{R}$ by

$$E[v] = \frac{1}{2} \int_\Omega \nabla v \cdot (A \nabla v) \, d\boldsymbol{x} - \int_\Omega f v \, d\boldsymbol{x},$$

where $f : \Omega \to \mathbb{R}$ is smooth and $A : \Omega \to \mathbb{R}^{2\times 2}$ is a smooth, matrix–valued map such that $A(\boldsymbol{x})$ is symmetric for all $\boldsymbol{x} \in \Omega$. Assume that $u \in C^2(\overline{\Omega}) \cap V$ minimises E:

$$E[u] = \min_{v \in V} E[v].$$

Show that u satisfies the elliptic PDE

$$-\mathrm{div}\,(A \nabla u) = f \quad \text{in } \Omega.$$

SECTION B

7. Let $a \in \mathbb{R}$ be a constant. Consider the first order PDE

$$\begin{cases} au_x + xyu_y = xu, & (x,y) \in \mathbb{R}^2, \\ u(0,y) = f(y), & y \in \mathbb{R}, \end{cases} \tag{3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous function.

(a) For which values of a is the initial condition in (3) noncharacteristic?

(b) Using the method of characteristics, solve (3) for all values of a for which the initial condition is noncharacteristic.

(c) Take $f(y) = y$, $a = 0$. Prove that (3) has infinitely many solutions.

8. Consider the scalar conservation law

$$\begin{cases} u_t + (f(u))_x = 0, \\ u(x,0) = u_0(x), \end{cases}$$

where

$$u_0(x) = \begin{cases} 1 & \text{for } x < 0, \\ 3 & \text{for } x > 0, \end{cases} \qquad \text{and} \qquad f(u) = \frac{1}{3}u^3.$$

(a) Write down the equation for the characteristics inside the regions

$$\{x < t\} \qquad \text{and} \qquad \{x > 9t\}.$$

(b) Verify that the following are weak solutions:

$$u_1(x,t) = \begin{cases} 1 & \text{for } x < \frac{7}{3}t, \\ 2 & \text{for } \frac{7}{3}t < x < 4t, \\ \left(\frac{x}{t}\right)^{1/2} & \text{for } 4t < x < 9t, \\ 3 & \text{for } x > 9t, \end{cases} \qquad u_2(x,t) = \begin{cases} 1 & \text{for } x < t, \\ \left(\frac{x}{t}\right)^{1/2} & \text{for } t < x < 9t, \\ 3 & \text{for } x > 9t. \end{cases}$$

(c) Which of the solutions in part (b) satisfy the Lax entropy condition? Justify your answer.

9. Let $f \in C_c^2(\mathbb{R}^3)$ and let
$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi|\boldsymbol{x}|}$$
be the fundamental solution of Poisson's equation in $\mathbb{R}^3$.

(a) Show that for all $\boldsymbol{x} \in \mathbb{R}^3$
$$\lim_{\varepsilon \to 0} \int_{B_\varepsilon(\mathbf{0})} \Phi(\boldsymbol{y}) \Delta_{\boldsymbol{y}} f(\boldsymbol{x} - \boldsymbol{y}) \, d\boldsymbol{y} = 0.$$
Hint: You may use the following, which you do not need to prove:
$$\|\Phi\|_{L^1(B_\varepsilon(\mathbf{0}))} = \frac{1}{2}\varepsilon^2.$$

(b) Show that for all $\boldsymbol{x} \in \mathbb{R}^3$
$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(\mathbf{0})} (\nabla_{\boldsymbol{y}} f(\boldsymbol{x} - \boldsymbol{y})) \cdot \boldsymbol{n}(\boldsymbol{y}) \, \Phi(\boldsymbol{y}) \, dS(\boldsymbol{y}) = 0$$
where $\boldsymbol{n}$ is the outward-pointing unit normal vector field to $\partial B_\varepsilon(\mathbf{0})$.

(c) Let $\boldsymbol{y} \in \partial B_\varepsilon(\mathbf{0})$. Show that
$$\nabla \Phi(\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y}) = -\frac{1}{4\pi\varepsilon^2}.$$

(d) Use part (c) to show that for all $\boldsymbol{x} \in \mathbb{R}^3$
$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^3 \setminus B_\varepsilon(\mathbf{0})} \nabla \Phi(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} f(\boldsymbol{x} - \boldsymbol{y}) \, d\boldsymbol{y} = f(\boldsymbol{x}).$$
Hint: You may also use the following, which you do not need to prove:
$$\Delta \Phi(\boldsymbol{x}) = 0 \text{ for } \boldsymbol{x} \neq \mathbf{0}.$$

(e) Define $u : \mathbb{R}^3 \to \mathbb{R}$ by
$$u(\boldsymbol{x}) = \int_{\mathbb{R}^3} \Phi(\boldsymbol{y}) f(\boldsymbol{x} - \boldsymbol{y}) \, d\boldsymbol{y}.$$
Use parts (a), (b) and (d) to show that for all $\boldsymbol{x} \in \mathbb{R}^3$
$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}).$$
Hint: You may also use the following, which you do not need to prove:
$$\Delta u(\boldsymbol{x}) = \int_{\mathbb{R}^3} \Phi(\boldsymbol{y}) \Delta_{\boldsymbol{y}} f(\boldsymbol{x} - \boldsymbol{y}) \, d\boldsymbol{y}.$$

10. (a) Let $u : [0,1] \times [0,\infty) \to \mathbb{R}$ be a smooth solution of

$$\begin{aligned} u_t(x,t) - u_{xx}(x,t) &= 0 \quad \text{for } (x,t) \in (0,1) \times (0,\infty), \\ u(x,0) &= x(1-x) \quad \text{for } x \in (0,1), \\ u(0,t) = u(1,t) &= 0 \quad \text{for } t \in [0,\infty). \end{aligned}$$

(a1) Define $v(x,t) = x(1-x)e^{-\beta t}$. Find the largest value of $\beta_0 > 0$ such that for all $\beta \in (0, \beta_0]$

$$v_t - v_{xx} \geq 0 \quad \forall\, (x,t) \in (0,1) \times (0,\infty).$$

(a2) For all $\beta \in (0,\beta_0]$, prove that

$$0 \leq u(x,t) \leq x(1-x)e^{-\beta t} \quad \forall\, (x,t) \in (0,1) \times (0,\infty).$$

(b) Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary and $f : \Omega \to \mathbb{R}$ be smooth. Consider the semilinear elliptic PDE

$$\begin{aligned} -\Delta u + c(u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

where $c : \mathbb{R} \to \mathbb{R}$ is a smooth, non-decreasing function, i.e., $c(t) \geq c(s)$ if $t \geq s$. Prove that (4) has at most one smooth solution.

SECTION C

11. Recall that, for $a \in \mathbb{R}$, δ_a is the distribution defined by $(\delta_a, \varphi) = \varphi(a)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. Let $n \in \mathbb{N}$, and consider the sequence of functions $u_n : \mathbb{R} \to \mathbb{R}$ given by

$$u_n(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ nx & \text{for } 0 < x < 1/n, \\ 1 & \text{for } x \geq 1/n. \end{cases}$$

(a) Compute the distributional derivative u_n'.

(b) Prove that $u_n' \to \delta_0$ as $n \to \infty$, in the sense of distributions.

(c) Prove that $\sqrt{u_n'} \to 0$ as $n \to \infty$, in the sense of distributions.